Population Extinction on a Random Fitness Seascape

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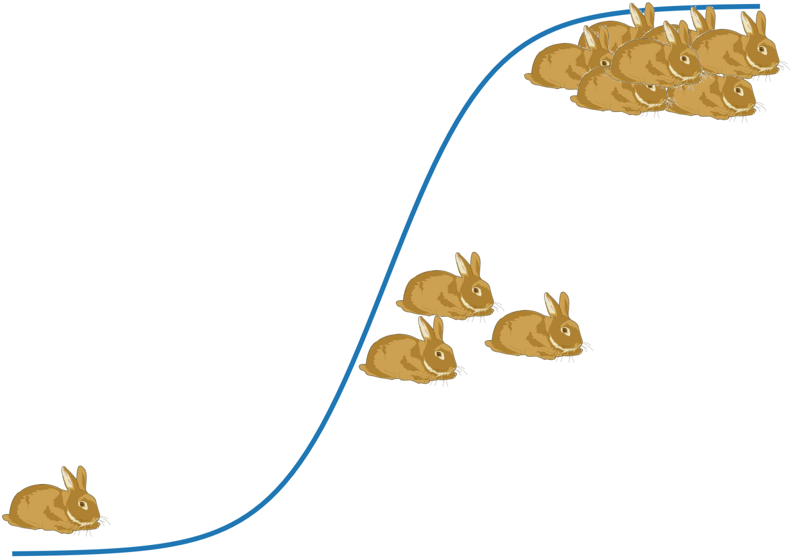
Population Growth



Why Are There So Many Population Models?

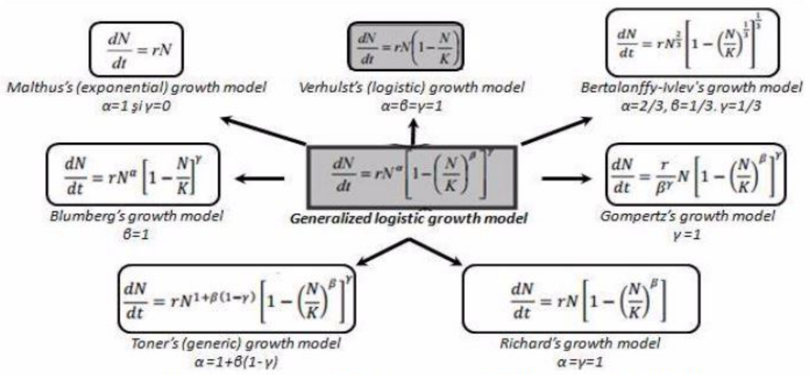


FIG. 3. The generalized logistic curve and its derivative models [11]

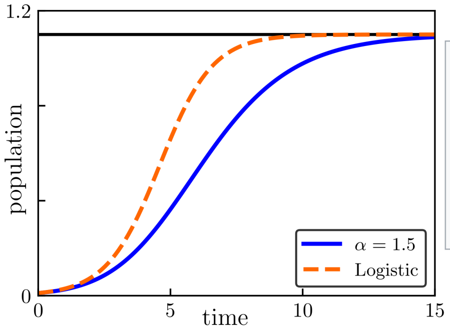
Figure: Cioruța (2016)

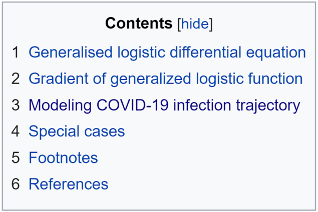
Richards' Equation

The **Richards' Equation** generalizes the logistic equation's second-order term, so

$$\partial_t y = \mu y - a y^{\boxed{\alpha}}.$$

This is commonly used in forestry and epidemics, but the value of α isn't always motivated.




Contents [hide]

A Spatial Population Model As An Origin

The Fisher Equation

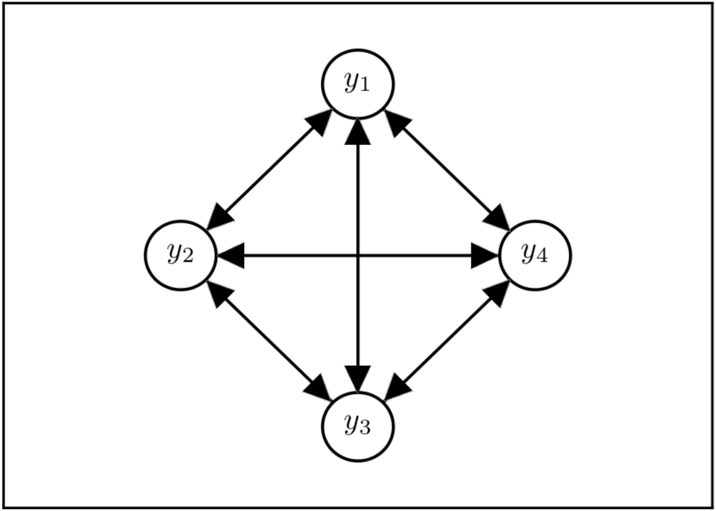
The **Mean-Field Fisher Equation** *is given by*

$$\partial_t y = \mu y - a y^2 + D(\bar{y} - y),$$

where

- μ *sets growth rate,*
- a *sets saturation population,*
- D *sets diffusion rate, and*
- $\bar{y}$ *is the spatial average.*

A Mean-Field Model



Fitness is Time Dependent

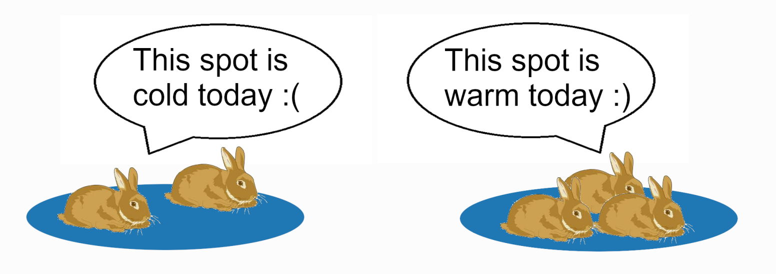
Figure: Merrell (1994)

Randomness in Fitness is Seascape Noise

We respresent seascape noise by treating the fitness parameter μ as noisy. So,

$$\mu \to \mu + \sigma\eta,$$

where η is a stochastic term with unit variance and zero mean.



Seascape Noise

The Mean-Field Seascape Fisher Equation

The Fisher Equation with seascape noise is written as

$$\partial_t y = \mu y - a y^2 + D(\bar{y} - y) + \boxed{\sigma y \eta},$$

where $y\eta$ *represents randomness in fitness and enviroment*

Stationary State First

Good News:

- If a stationary state exists, then the spatial average pop. $\bar{y}$ should be the same as the ensemble averaged pop. $\langle y \rangle$ via the Law of Large Numbers.

Bad News:

- In the absence of noise, the Fisher equation converges to $\bar{y} \to \mu/a$.

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- However, $\bar{y} \not\to \mu/a$ in the presence of noise.

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- In the absence of noise, the Fisher equation converges to $\bar{y} = \mu/a$.
- However, $\bar{y} \not\to \mu/a$ in the presence of noise.
- $\bar{y}$ must be obtained through self-consistency.

Self-Consistent

To be self consistent, we need

$$\partial_t \langle y \rangle = 0$$

Therefore, we need

$$\begin{aligned} 0 = & \langle \mu y - a y^2 + D(\bar{y} - y) + \sigma y \eta \rangle \\ = & \mu \bar{y} - a \langle y^2 \rangle \end{aligned}$$

So our condition is

$$\boxed{\mu \bar{y} = a \langle y^2 \rangle}$$

Distribution Solution

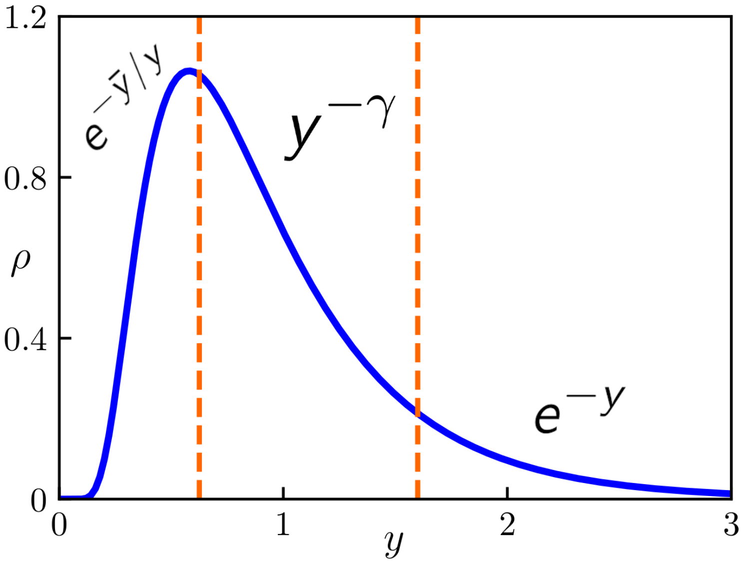
Distribution in the Growth Case

When $\mu > 0$, the population distribution under seascape noise is proportional to

$$\hat{\rho}(y, \bar{y}) = e^{-c_D \bar{y}/y} y^{-2-c_D+c_\mu} e^{-c_a y} \ .$$

where $c_D = 2D/\sigma^2$, $c_\mu = 2\mu/\sigma^2$, and $c_a = 2a/\sigma^2$, and $\bar{y}$ is chosen self-consistently.

$$\hat{\rho}(y, \bar{y}) = e^{-\bar{y}/y} y^{-\gamma} e^{-y}$$



Self-Consistent mean

Using the cutoffs in $\hat{\rho}$, we can estimate

$$\mu \bar{y} = a \langle y^2 \rangle = a \frac{\int_0^\infty y^2 \hat{\rho}(y, \bar{y}) dy}{\int_0^\infty \hat{\rho}(y, \bar{y}) dy},$$

and solve the self-consistency condition.

Mean Pop at Small Fitness

Scaling Law for Seascape Noise

In the extinction limit, the population mean scales with the fitness as:

$$\bar{y} \propto \begin{cases} \mu, & 2D > \sigma^2; \\ \mu^{\sigma^2/(2D)}, & 2D < \sigma^2. \end{cases}$$

Notice that when $2D/\sigma^2 < 1$, we have an anomalous scaling exponent.

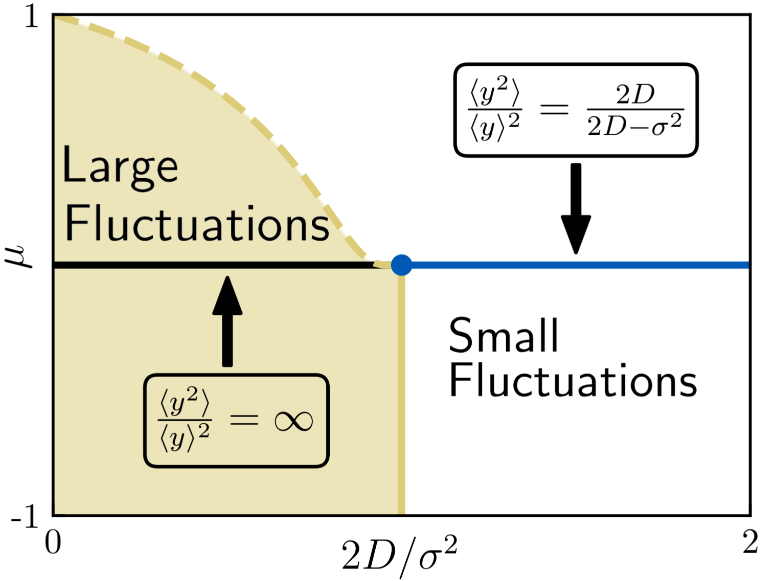
Moment Ratio with $\mu > 0$

Because we know the correct value of $\bar{y}$, we can calculate higher moments, and find that

$$\frac{\langle y^2 \rangle}{\langle y \rangle^2} = \frac{2D}{2D - \sigma^2}, \qquad 2D > \sigma^2 \ ;$$
$$\frac{\langle y^2 \rangle}{\langle y \rangle^2} \propto \mu^{-\sigma^2/(2D)+1}, \quad 2D < \sigma^2 \ .$$

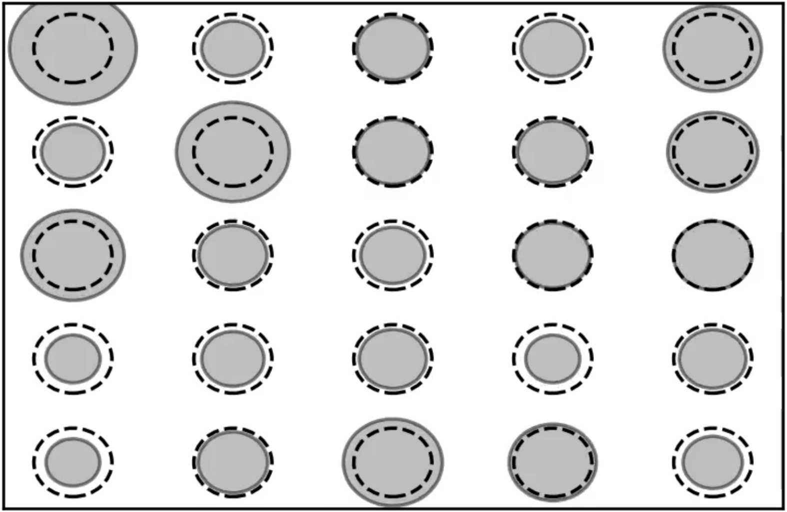
So as μ gets small, the variance may or may not get large compared to the mean.

Phase Diagram for Seascape Noise



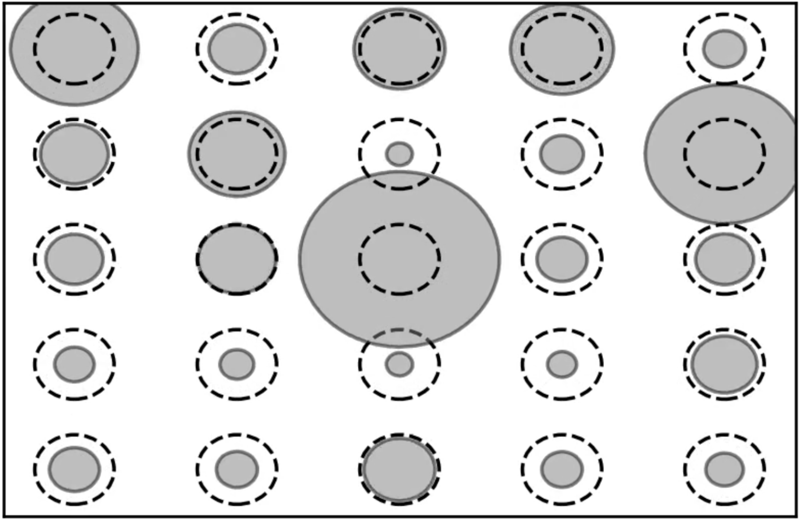
High D $\implies$ Low Variance

$$c_\mu = c_a = 0, c_D = 4.0, \sigma = 1.0, t = 5.0$$



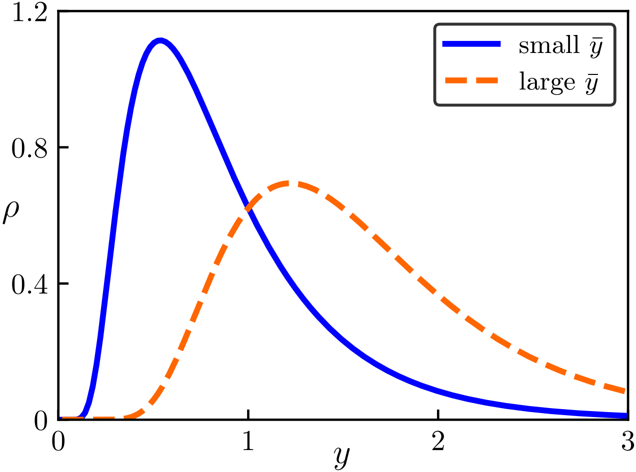
Low D $\Longrightarrow$ High Variance

$$c_\mu = c_a = 0, c_D = 0.5, \sigma = 1.0, t = 5.0$$



Taking Stationary Solution into Dynamics....

Assume the distribution still looks like $\hat{\rho}$, but for a time-varying $\bar{y}$...



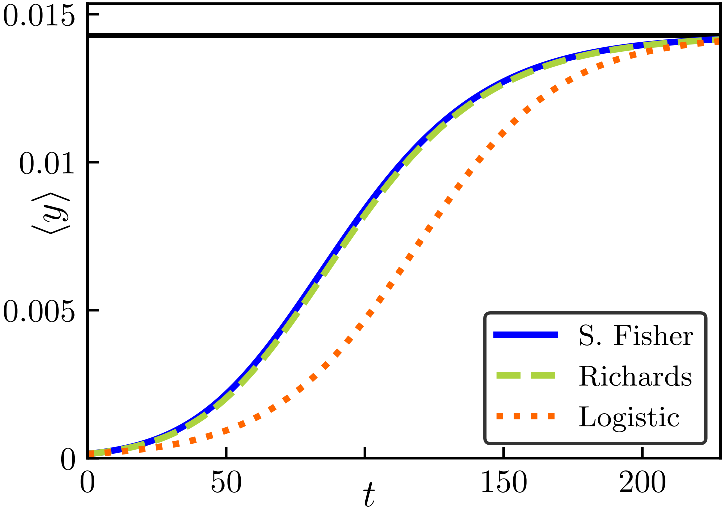
Fisher Averaged leads to Richards' Equation

Near extinction, the mean of the Stochastic Fisher Eq. behaves as

$$\begin{aligned}\partial_t \langle y \rangle &= \mu \langle y \rangle - a \langle y^2 \rangle + D \langle \bar{y} - y \rangle + \langle \sigma y \eta \rangle \\ &= \mu \langle y \rangle - a \langle y^2 \rangle \\ &\approx \mu \langle y \rangle - a \bar{y}^{1+\beta},\end{aligned}$$

which is a Richards' Equation!

Dynamics of the Mean



Final Summary

Seascape Noise + Spatial Diffusion

$\Downarrow$

Anomalous, Nonuniversal Power Laws

$\Downarrow$

Richards' Equation!

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Questions?

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Different Kinds of Noise

Tails	y small	y intermediate	y large
Seascape	$\exp\left[1/y\right]$	$y^{-\eta}$	$\exp\left[-y\right]$
Demographic	$y^{-\eta}$	$\exp[-y]$	$\exp\left[-y^2\right]$
Mixed	$y^{-\eta}$	$(y+r)^{-\gamma}$	$\exp\left[-y\right]$