A Seascape Origin of Richards Growth

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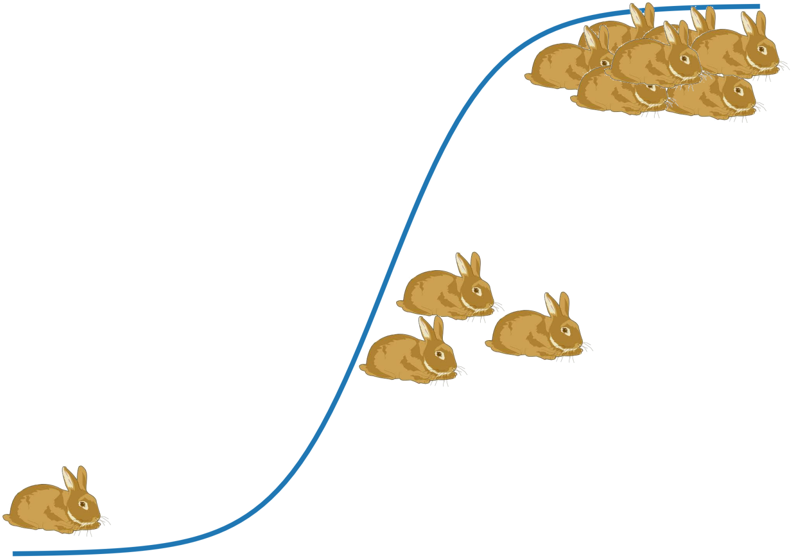
Population Growth



Why Are There So Many Population Models?

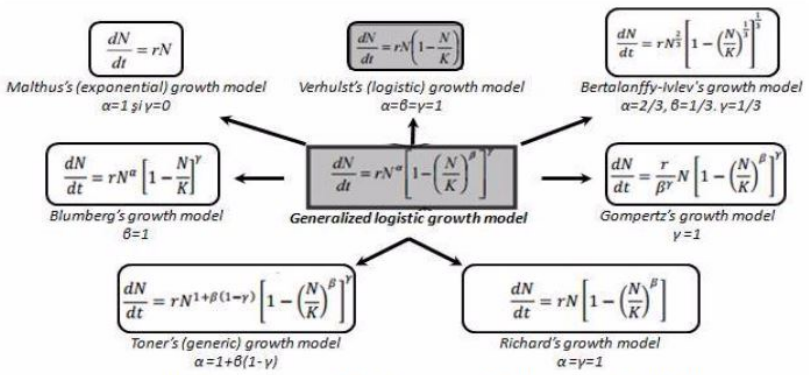


FIG. 3. The generalized logistic curve and its derivative models [11]

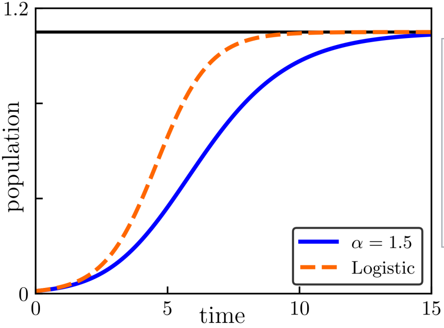
Figure: Cioruța (2016)

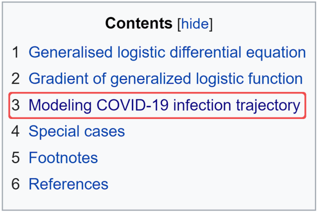
Richards' Equation

The **Richards' Equation** generalizes the logistic equation's second-order term, so

$$\partial_t y = \mu y - a y^{\boxed{\alpha}}.$$

This is commonly used in forestry and epidemics, but the value of α isn't always motivated.




Contents [hide]

Where Does It Come From?

Previous explainations have been to...

- Do a detailed manipulation of SIR, or
- Assume a fractal spatial struture,

neither of which are particularly general.

How about a Spatial Population Model?

The Fisher Equation

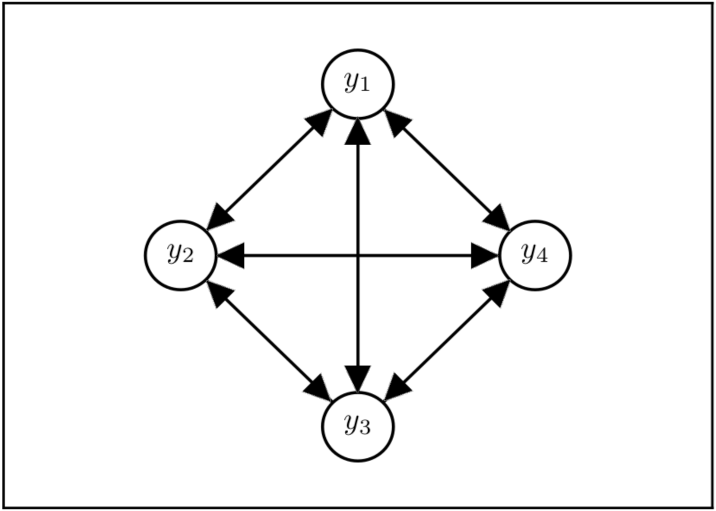
The **Mean-Field Fisher Equation** *is given by*

$$\partial_t y = \mu y - a y^2 + D(\bar{y} - y),$$

where

- μ *sets growth rate,*
- a *sets saturation population,*
- D *sets diffusion rate, and*
- $\bar{y}$ *is the spatial average.*

A Mean-Field Model



Fitness is Time Dependent

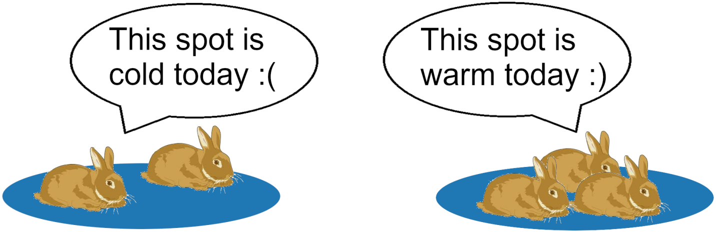
Figure: Merrell (1994)

Randomness in Fitness is Seascape Noise

We respresent seascape noise by treating the fitness parameter μ as noisy. So,

$$\mu \to \mu + \sigma\eta,$$

where η is a stochastic term with unit variance and zero mean.



Seascape Noise

The Mean-Field Seascape Fisher Equation

The Fisher Equation with seascape noise is written as

$$\partial_t y = \mu y - a y^2 + D(\bar{y} - y) + \boxed{\sigma y \eta},$$

where $y\eta$ represents randomness in fitness and enviroment

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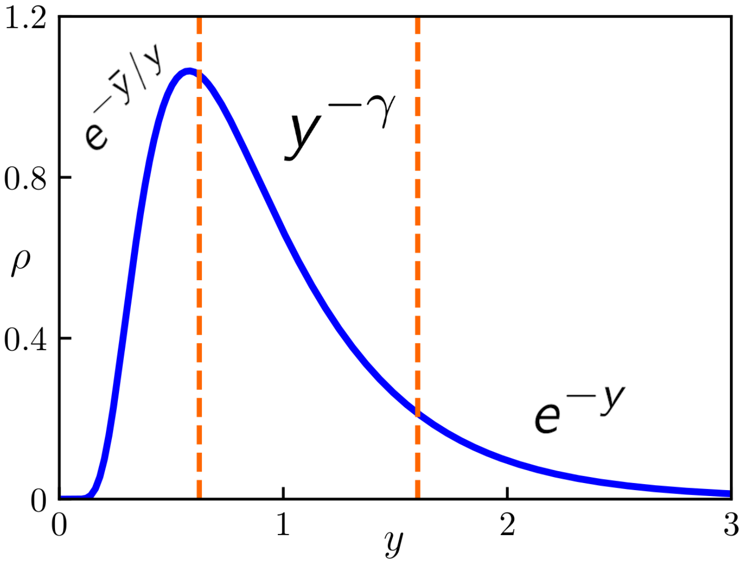
- This has a known steady-state solution!

$\hat{\rho}(y, \bar{y}) = e^{-\bar{y}/y} y^{-\gamma} e^{-y}$



How Do We Make The Stationary Solution into Dynamics?

Dynamics?

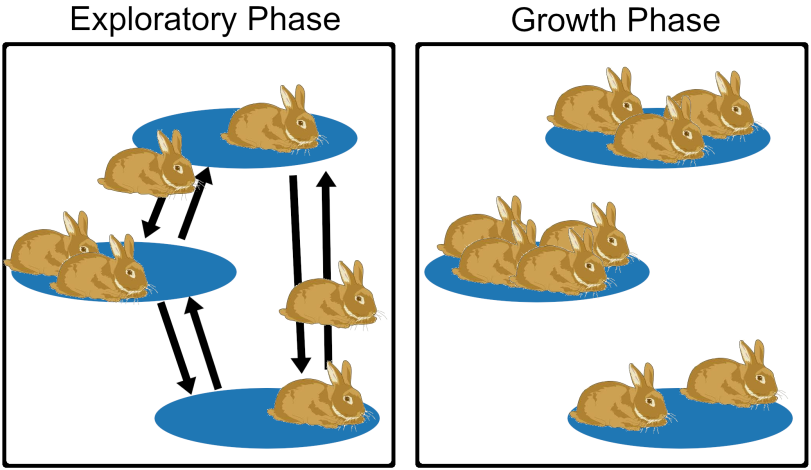
$$\partial_t y = \boxed{\mu y - a y^2} + \boxed{D(\bar{y} - y) + \sigma y \eta}$$

- Simultaneous growth and exploration is difficult to analyze...

Just Split Them!

- Explore, then grow, then explore, then grow, and so on, and so on....

Seasonal Growth Model



Explore, Grow, Repeat...

Exploratory Phase Dynamics

$$\partial_t y = D(\bar{y} - y) + \sigma y \eta$$

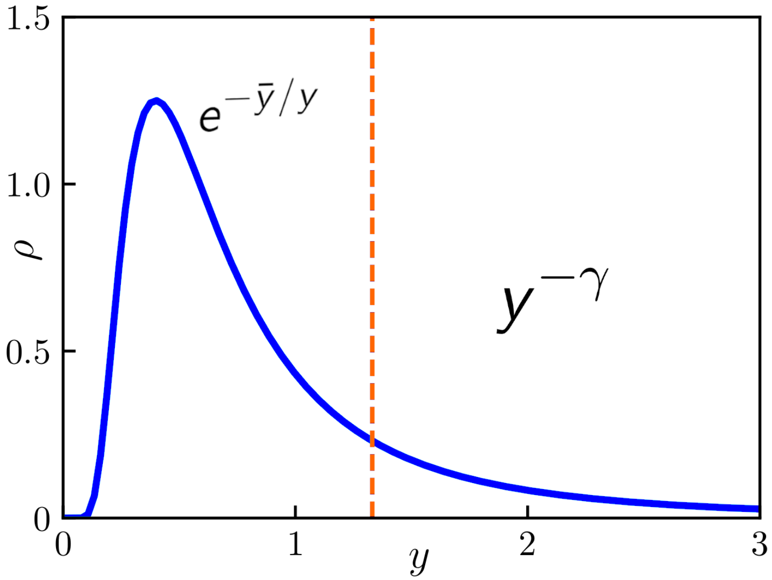
$\Longrightarrow$ *Diffusion with seascape noise.*

Growth Phase Dynamics

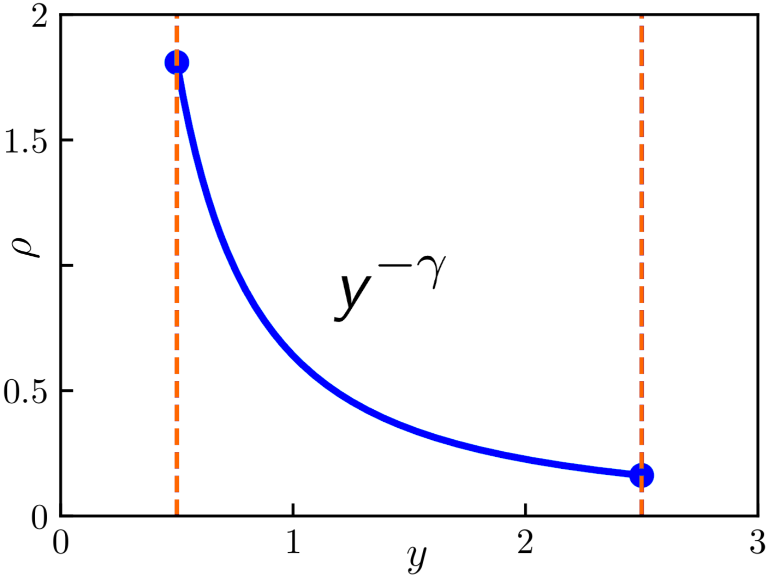
$$\partial_t y_i = \mu y_i - a {y_i}^2$$

$\Longrightarrow$ *Logistic growth*

Problem With Exploring: No Upper Cutoff



Time is Finite, so Simplify



Time is Finite, so Simplify

Given this, we expect

$$\langle y^2 \rangle \propto \langle y \rangle^{1+2D/\sigma^2},$$

leading to a spatially dependant power law[1].

[1] So long as $2D < \sigma^2$

Logistic Growth

The growth phase is deterministic, so each node goes like

$$y_i(t) = \frac{\mu y_i e^{\mu t}}{a y_i (e^{\mu t} - 1) + \mu}.$$

Short Growth Phase

If the timescale of growth Δt is small, then we can average over the exporation steady state and Taylor expand:

$$\langle y(t + \Delta t) \rangle = (1 + \mu \Delta t) \langle y(t) \rangle - a \langle y^2(t) \rangle \Delta t.$$

Short Growth Phase

If the upper cutoff Λ in the exporatory phase is sufficiently large compared to its mean, then

$$\frac{\langle y(t) + \Delta t\rangle - \langle y(t)\rangle}{\Delta t} = \mu\langle y(t)\rangle - \tilde{a}(\Lambda)\langle y(t)\rangle^{\gamma} + O\left((\langle y(t)\rangle/\Lambda)^{2D/\sigma^2}\right)$$

Combined Dynamics

Combining the two phases leads to an averaged dynamics of

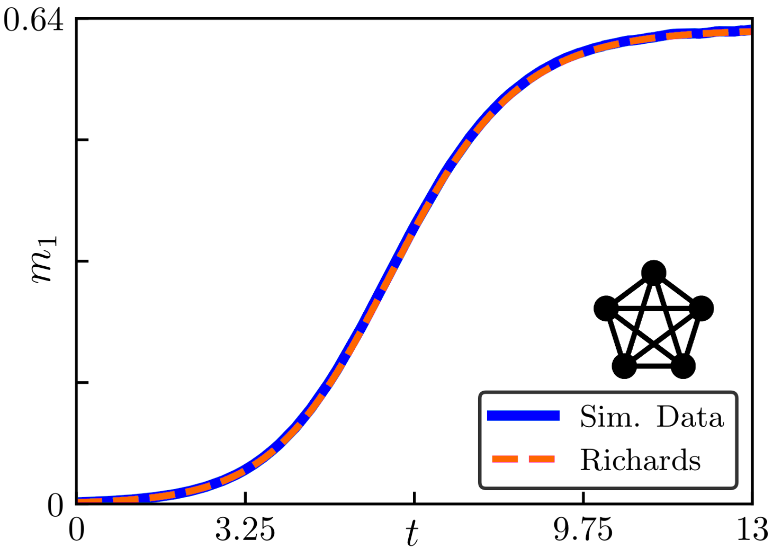
$$\partial_t \langle y \rangle = \tilde{\mu} \langle y \rangle - \tilde{a} \langle y \rangle^{\gamma},$$

where $\tilde{\mu}$ and $\tilde{a}$ depend on the time spent in each phase.

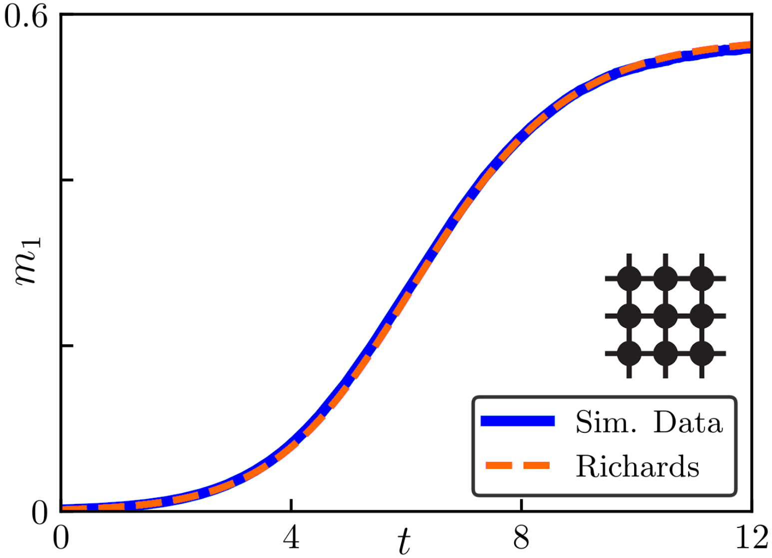
An Origin of Richards Growth

1. Diffusion under seascape noise produces power law distributions in space.
2. Under logistic growth, this power law shows up in the dynamics of the averages.

Mean Field



Shows up for 2D too



Universality

- Because the derivation is insentivie to perturbation in the growth phase, we get **Universality** that other approaches lack.

Final Summary

Seascape Noise + Spatial Diffusion

$\Downarrow$

Anomalous, Nonuniversal Power Laws

$\Downarrow$

Richards' Equation!

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Questions?

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Seascape origin of Richards growth

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First proposed as an empirical rule over half a century ago, the Richards growth equation has been frequently

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