Evaluating the Logic of Selection in Germinal Centers

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What is a Germinal Center?

GC: Keystone of Adaptive Immunity

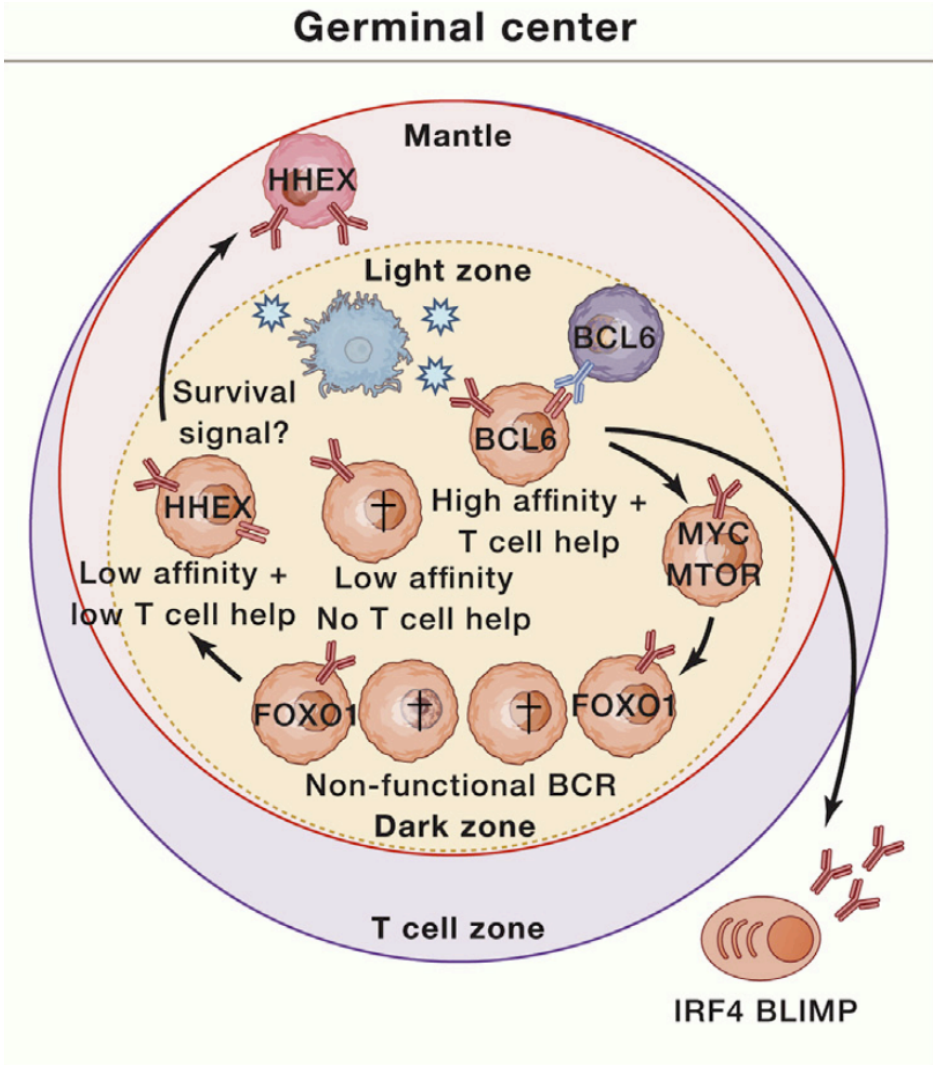


Figure: Young (2021)

GC: Small Scale Evolutionary Optimization Alg.

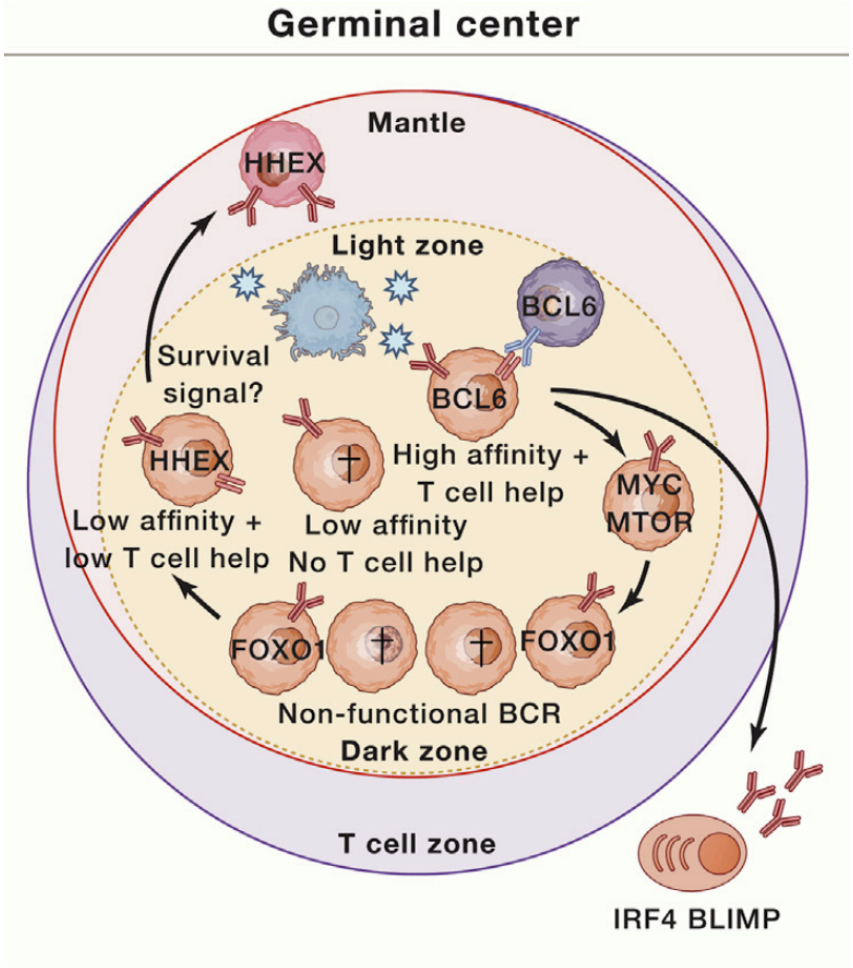


Figure: Young (2021)

- Body gets hit by pathogen.
- Anitgen gets presented to B-Cells in Germinal Center.
- Speedy evolution for **Affinity** (binding ability).
- Get high quality antibodies (yay!)

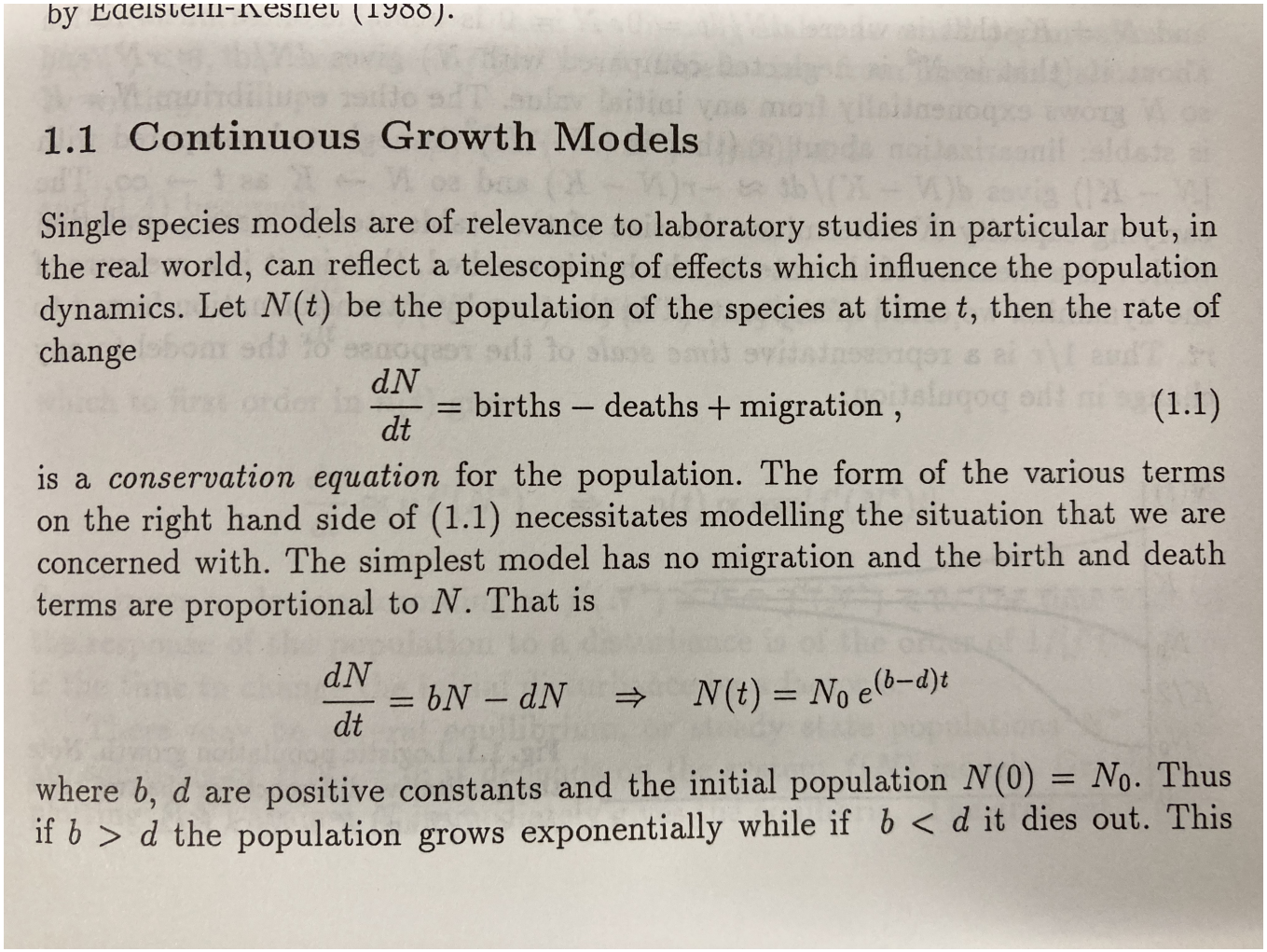
Question: How *Does* Selection Take Place?

Standard Models: Single Parameter

Traditional population growth model:

$$\frac{dN}{dt} = (\text{birth rate} - \text{death rate})N$$

Normal models: Single Parameter



by Edelstein-Keshet (1988).

1.1 Continuous Growth Models

Single species models are of relevance to laboratory studies in particular but, in the real world, can reflect a telescoping of effects which influence the population dynamics. Let $N(t)$ be the population of the species at time t, then the rate of change

$$\frac{dN}{dt} = \text{births} - \text{deaths} + \text{migration}\,, \tag{1.1}$$

is a *conservation equation* for the population. The form of the various terms on the right hand side of (1.1) necessitates modelling the situation that we are concerned with. The simplest model has no migration and the birth and death terms are proportional to N. That is

$$\frac{dN}{dt} = bN - dN \quad \Rightarrow \quad N(t) = N_0\, e^{(b-d)t}$$

where b, d are positive constants and the initial population $N(0) = N_0$. Thus if $b > d$ the population grows exponentially while if $b < d$ it dies out. This

Figure: Murray (1993)

Birth and Death Are Not The Same

Birth Selection: High fitness = divide faster

Death Selection: High fitness = die slower

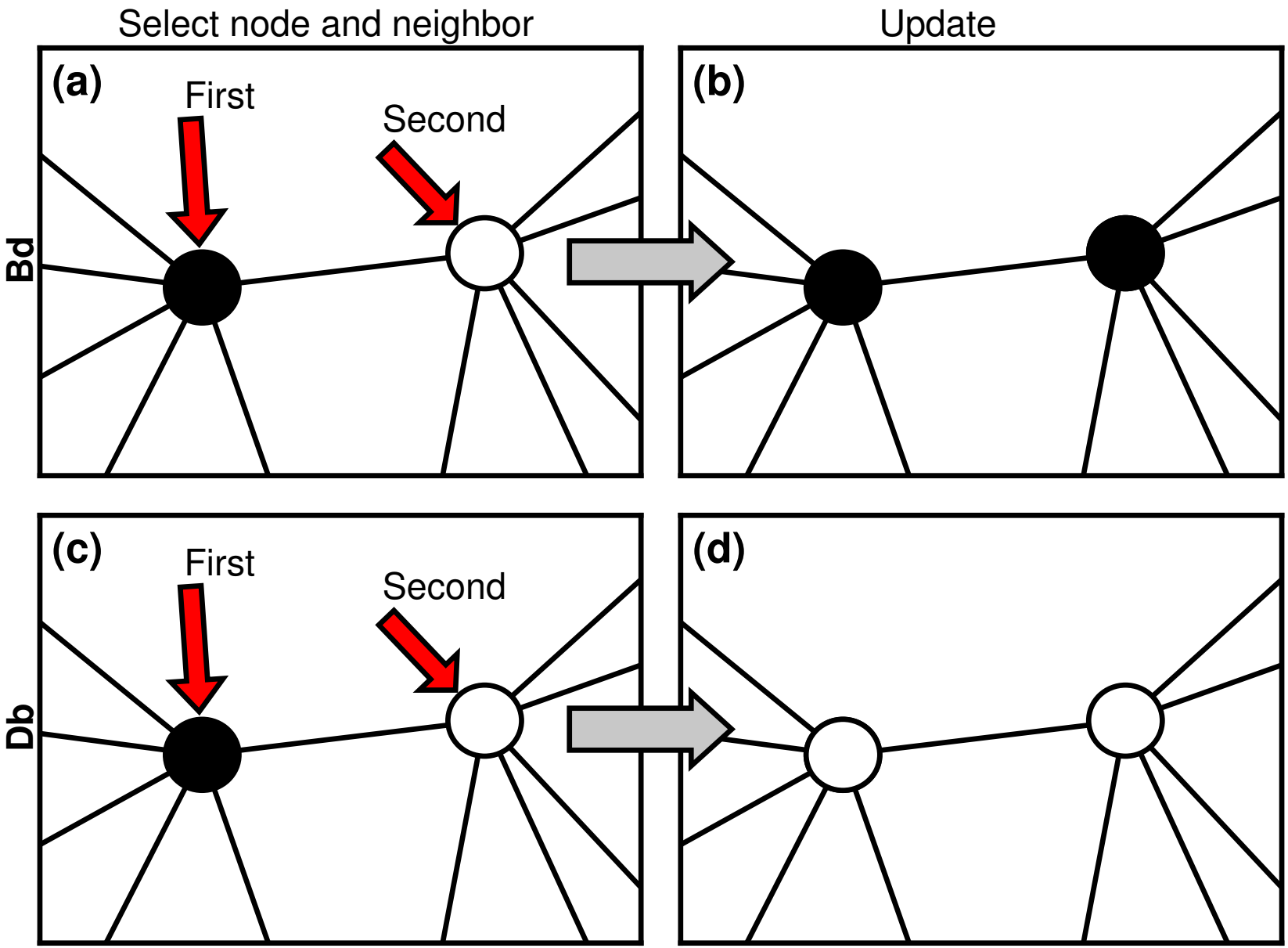
Previous Work: The Moran Model

Definition

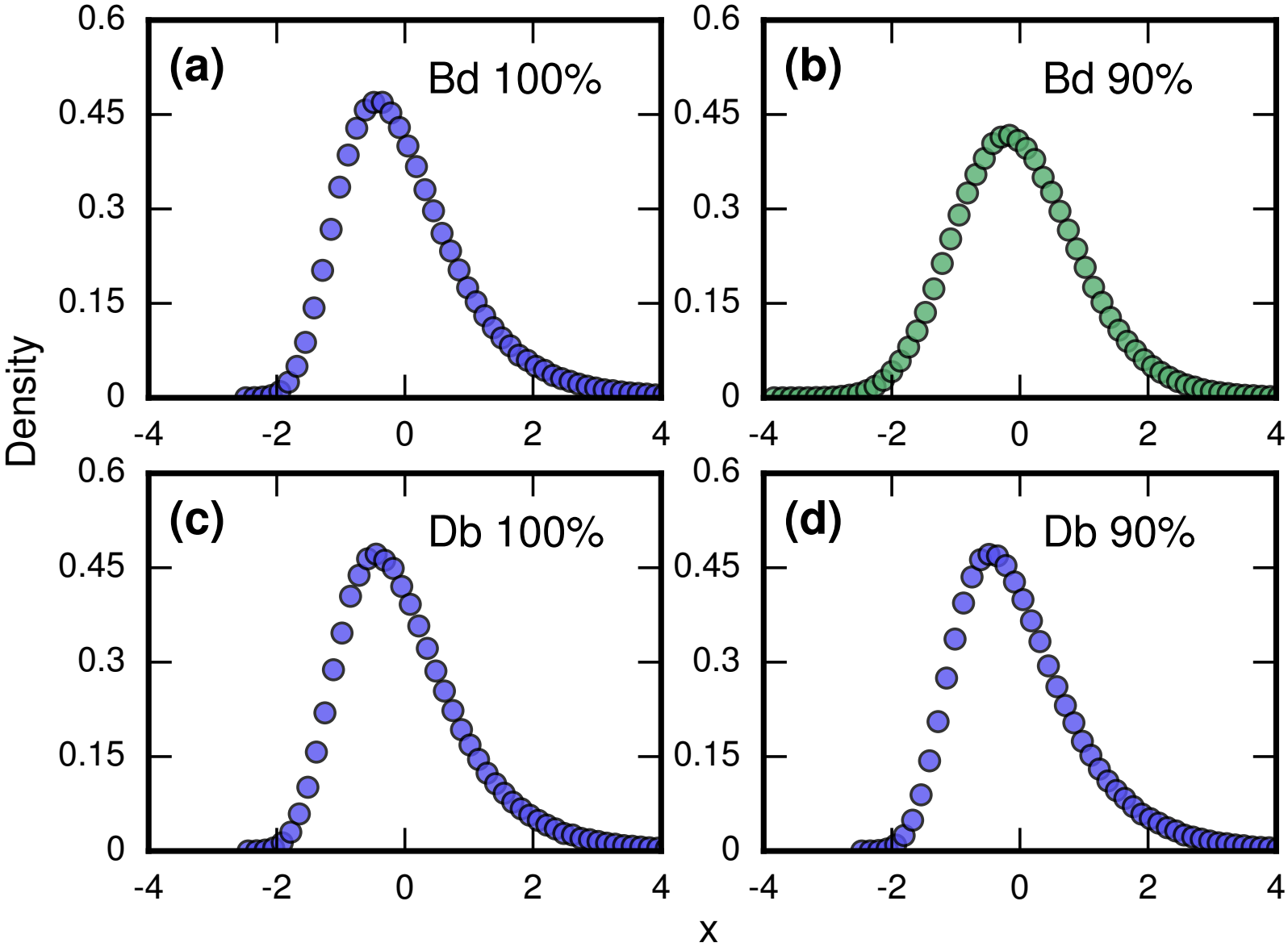
The **Moran Birth-death (Bd) model** consists of three steps:

1. With probability proportional to fitness (r), randomly select a cell to give birth.
2. Uniformly randomly select a neighbor of the first cell to die.
3. The dying cell takes on the type of the birthing cell.

Moran Model


Select node and neighbor
Update
(a)
First
Second
Bd
(b)
(c)
First
Second
Db
(d)

Fixation Times Distinguish Birth and Death



Birth and Death Are Not The Same

Intuition:

- Birth Selection slows down near total fixation.
- Death Selection starts slow, but fixates fast.
- Truncation skip slow regime for Birth Selection, changing the qualitative shape.

The Goal

Can we find a convenient signature of Death Selection in Germinal Centers?

A Model for Affinity Selection

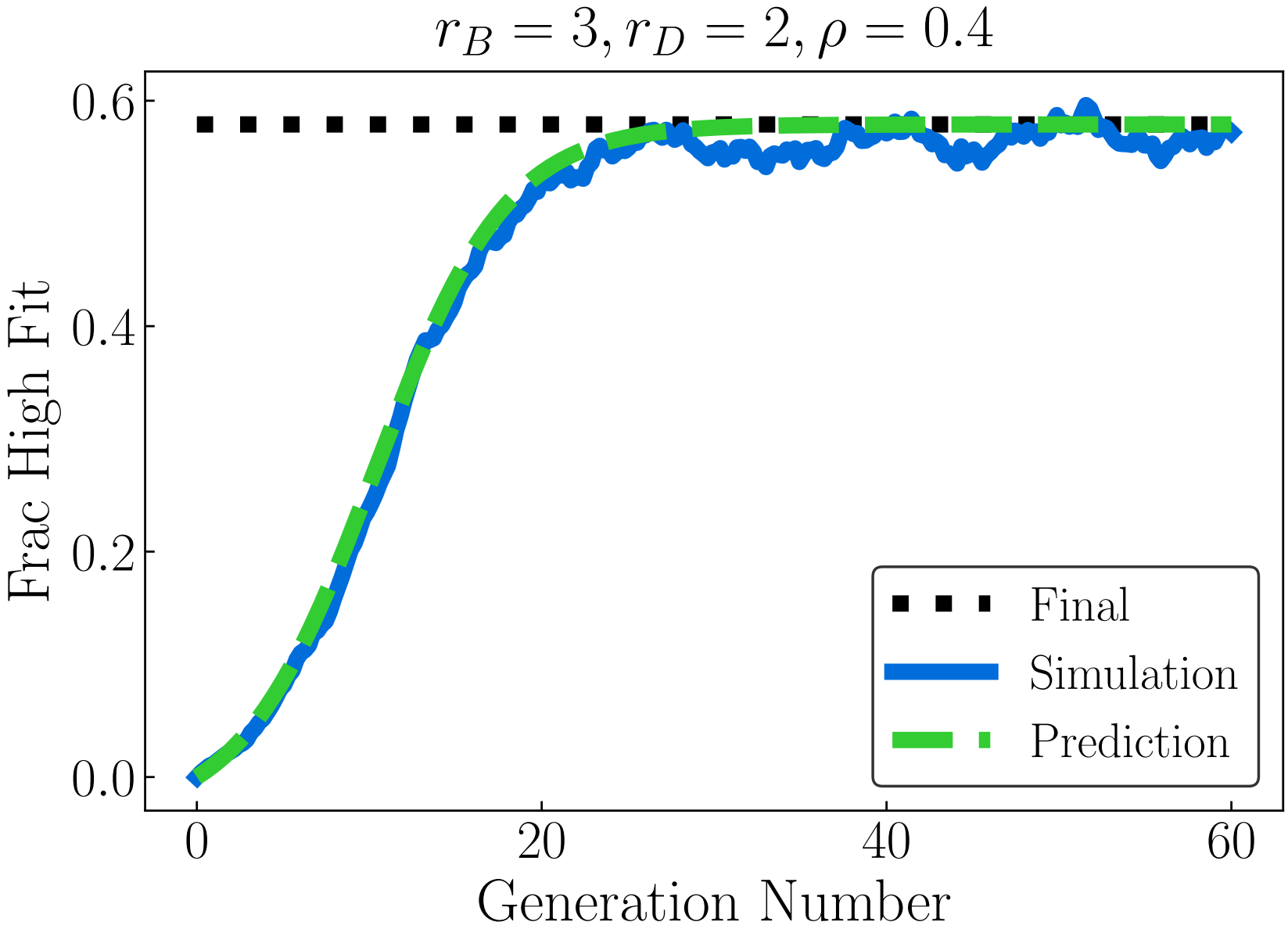
Only two affinities, High (H) and Low (L) for maximum disparity.

h = fraction of population which is high affinity.

GC has three potential selection mechanisms:

- Birth Selection (r_B),
- Death Selection (r_D),
- Mutational Selection (??).

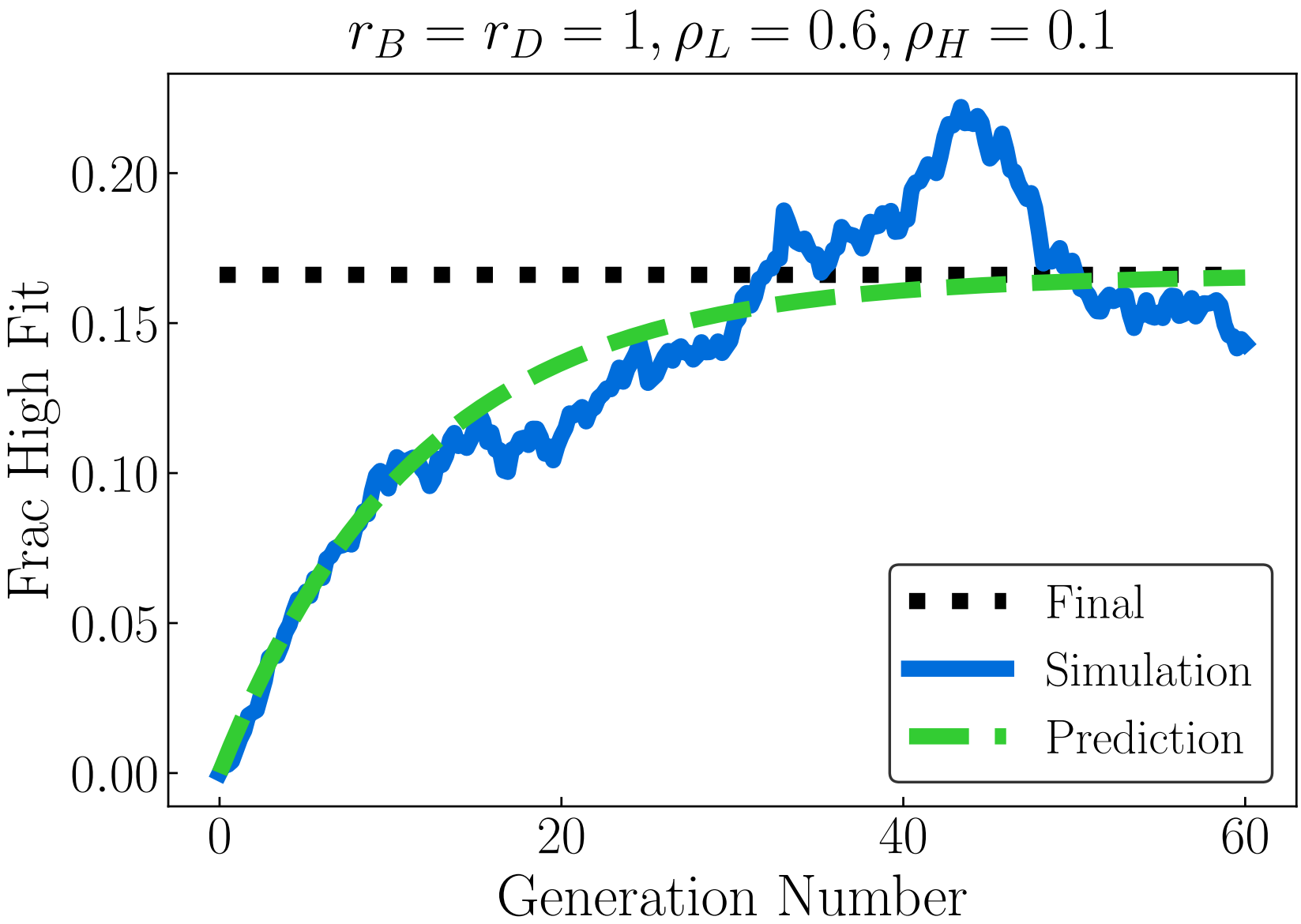
Typical Selection Curve



Mutational Selection?

- Mutations in GC occur 10^6 times more often than default.
- GCs occasionally have **Clonal Bursts**, where one cell line repeatedly divides.
- When bursting, the line has a **notably lower** mutation rate than normal for the GC.
- This opens the possibility of a **tunable mutation rate**.

Pure Mutational Selection



Average Selection

Let $\ell = 1 - h$, then the average selection dynamics are given by:

$$\boxed{\frac{dh}{dt} = \frac{2\rho_L\eta_L}{1+n/N}\frac{\ell}{r_B h+\ell} + \frac{1-2\rho_H\eta_H}{1+n/N}\frac{r_B h}{r_B h+\ell} - \frac{n/N}{1+n/N}\frac{h/r_D}{h/r_D+\ell}}$$

Skew Hypothesis

Skew Hypothesis

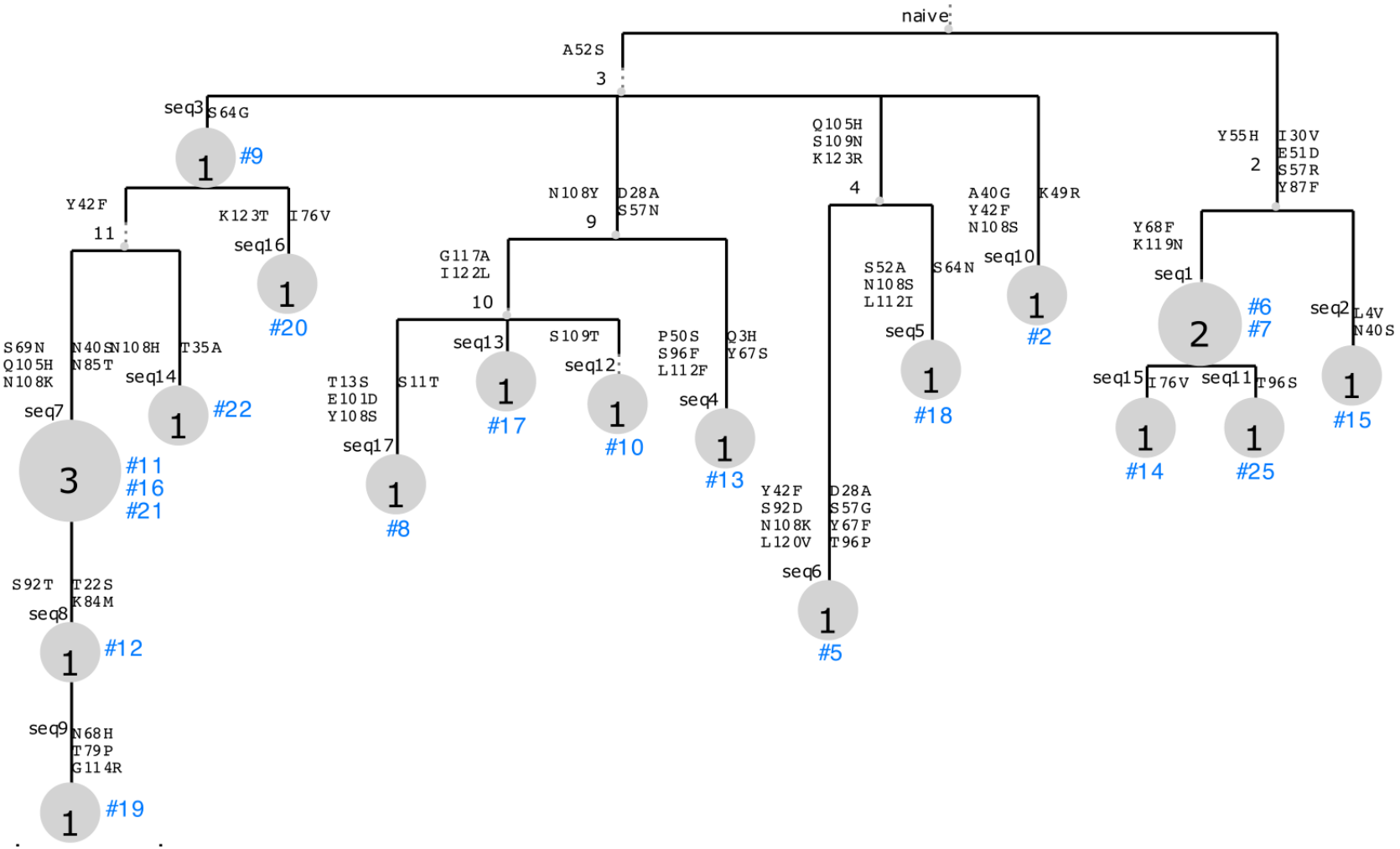
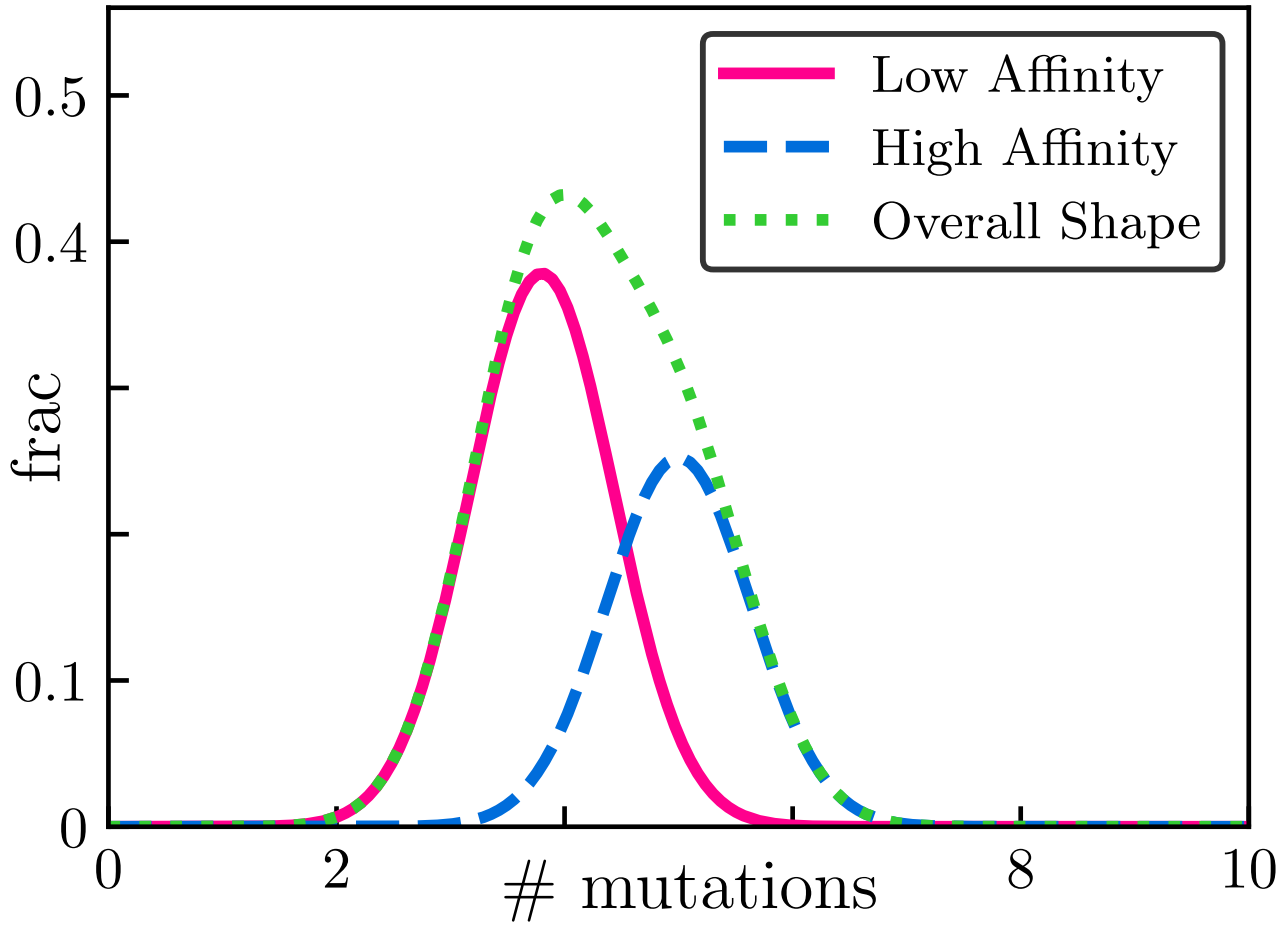


Figure: Tatsuya (2022)

Skew Hypothesis

Since L and H have different levels of mutational activity, would they seperate?



Calculation Outline: Mutation Dynamics

$$h_m = \frac{\text{number of H cells with m mutations}}{\text{number of cells}}$$

$$\ell_m = \frac{\text{number of L cells with m mutations}}{\text{number of cells}}$$

$$\begin{aligned} \langle m^k \rangle &= \text{k'th moment of mutation distribution} \\ &= \sum_m (h_m + \ell_m)\, m^k \end{aligned}$$

Calculation Outline: Linear Structure

$$\partial_t \langle m_H^k \rangle = A_H \langle m_H^k \rangle + B_H \sum_{w=0}^{k} \binom{k}{w} \langle m_H^k \rangle + C_H \sum_{w=0}^{k} \binom{k}{w} \langle m_L^k \rangle$$

$$\partial_t \langle m_L^k \rangle = A_L \langle m_L^k \rangle + B_L \sum_{w=0}^{k} \binom{k}{w} \langle m_H^k \rangle + C_L \sum_{w=0}^{k} \binom{k}{w} \langle m_L^k \rangle$$

Calculation Outline: Problem

$$\partial_t \vec{m} = Q\vec{m} + \vec{v}$$

Truncating to the third moment gives:

$$Q = \begin{pmatrix} J & 0 & 0 \\ 2R & J & 0 \\ 3R & 3R & J \end{pmatrix}$$

Calculation Outline: Problem

$$\partial_t \vec{m} = Q\vec{m} + \vec{v}$$

$$\Downarrow$$

$$\vec{m}(t) = -Q^{-1}\vec{v} + e^{Qt}\left(Q^{-1}\vec{v} + \vec{m}(0)\right)$$

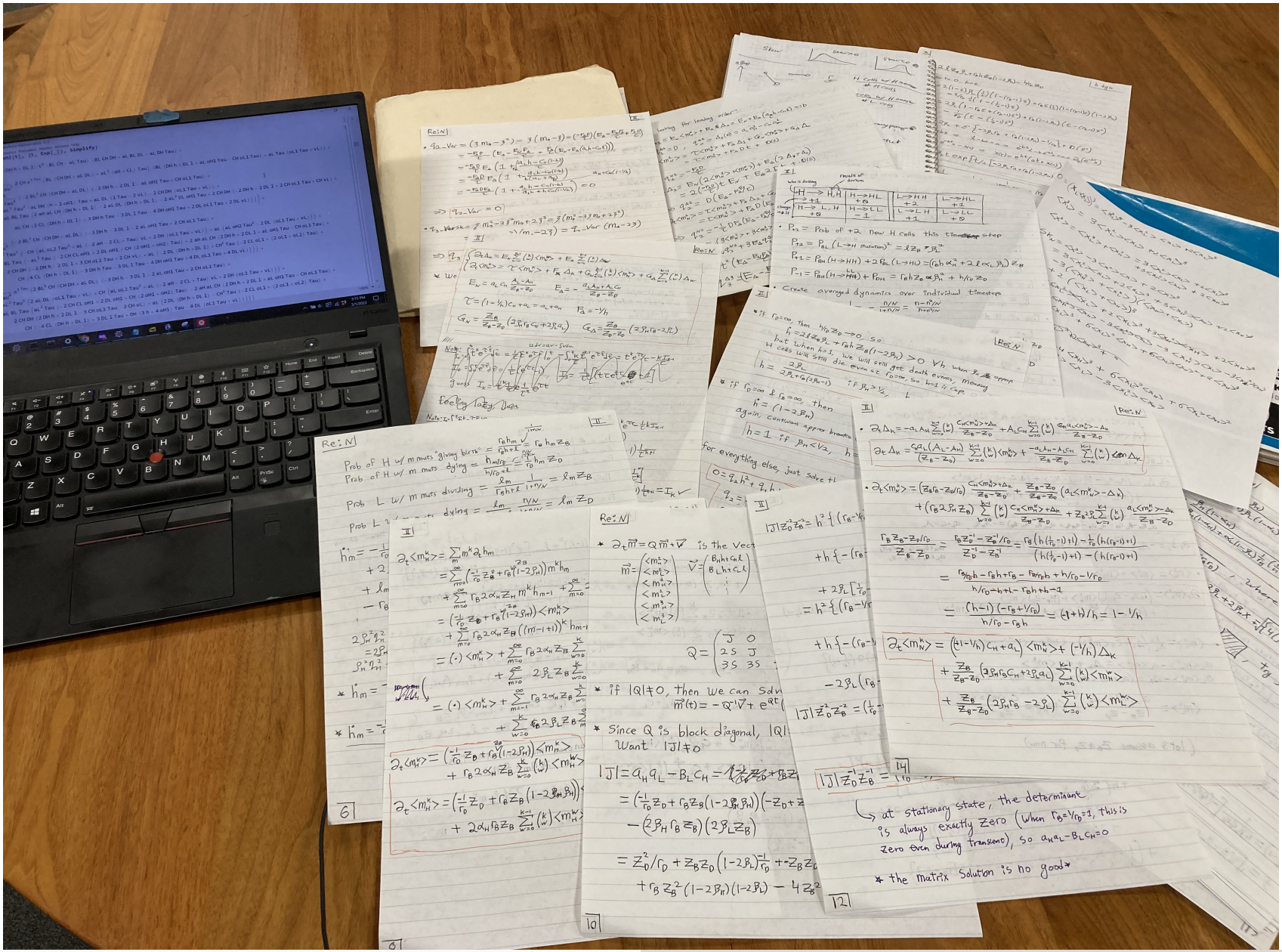
Calculation Outline: Back of Envelope

$$\text{Skew} = \frac{\text{Sk}}{\text{Var}^{3/2}}$$

$$\text{Sk} \approx \langle m^3 \rangle = \mathbb{O}(t^3)$$
$$\text{Var} \approx \langle m^2 \rangle = \mathbb{O}(t^2)$$

$$\text{Therefore Skew} \rightarrow \text{Constant}$$

N O P E


Re: N
Prob of H w/ m mates giving birth = $\frac{r_B h_m}{r_B h + 1} = r_B h_m z_B$
Prob of H w/ m mates dying = $\frac{h_m r_D}{h r_D + 1} \approx r_D h_m z_D$
Prob L w/ m mates dividing = $\frac{\lambda_m}{r_B h + 1} \frac{1}{1 + r_D N} = \lambda_m z_B$
Prob L w/ m mates dying = $\frac{l_m}{1 + r_D N} = l_m z_D$
$\dot{h}_m = -\frac{1}{r_D} h_m + \lambda_{m-1} z_B h_{m-1} - \lambda_m z_B h_m + r_B 2\alpha_{m+1} z_B h_{m+1} - r_B 2\alpha_m z_B h_m$
$\dot{h}_m = -\frac{1}{r_D} h_m + \lambda_{m-1} z_B h_{m-1} - \lambda_m z_B h_m + r_B 2\alpha_{m+1} z_B h_{m+1} - r_B 2\alpha_m z_B h_m$
$\partial_t \langle m_H^k \rangle = \sum_m m^k \partial_t h_m$
$= \sum_{m=0}^{\infty} \left(-\frac{1}{r_D} z_B^0 + r_B^0 (1-2\beta_H)\right) m^k h_m + \sum_{m=0}^{\infty} r_B 2\alpha_H z_H m^k h_{m-1} + \sum_{m=0}^{\infty} \ldots$
$= \left(\frac{1}{r_D} z_B + r_B^0 (1-2\beta_H)\right) \langle m_H^k \rangle + \sum_{m=0}^{\infty} r_B 2\alpha_H z_B \left((m-1+1)\right)^k h_{m-1}$
$= (\cdot) \langle m_H^k \rangle + \sum_{m=0}^{\infty} r_B 2\alpha_H z_B \sum_{w=0}^{k} \ldots$
$= (\cdot) \langle m_H^k \rangle + \sum_{m=-1}^{\infty} r_B 2\alpha_H z_B \sum_{w=0}^{k} \ldots + \sum_{w=0}^{k} r_B 2\beta_L z_B \ldots$
$\partial_t \langle m_H^k \rangle = \left(\frac{-1}{r_D} z_D + r_B^0 (1-2\beta_H)\right) \langle m_H^k \rangle + r_B 2\alpha_H z_B \sum_{w=0}^{k} \binom{k}{w} \langle m_H^w \rangle$
$\partial_t \langle m_H^k \rangle = \left(\frac{-1}{r_D} z_D + r_B z_B \left(1 - 2\beta_H r_B z_B\right)\right) + 2\alpha_H r_B z_B \sum_{w=0}^{k-1} \binom{k}{w} \langle m_H^w \rangle$
Re: N
$\partial_t \vec{m} = Q \vec{m} + \vec{V}$ is the Vec
$\vec{m} = \begin{pmatrix} \langle m_H^1 \rangle \\ \langle m_L^1 \rangle \\ \langle m_H^2 \rangle \\ \langle m_H^1 m_L^1 \rangle \\ \langle m_L^2 \rangle \\ \langle m_H^3 \rangle \\ \langle m_L^3 \rangle \end{pmatrix}$ $\vec{V} = \begin{pmatrix} B_H h_t c_H L \\ B_L h_t c_L L \\ \vdots \end{pmatrix}$
$Q = \begin{pmatrix} J & 0 & \\ 2S & J & \\ 3S & 3S & \ddots \end{pmatrix}$
* if $|Q| \neq 0$, then we can Sdv
$\vec{m}(t) = -Q^{-1}\vec{V} + e^{Qt}(\ldots)$
* Since Q is block diagonal, $|Q|$
Want $|J| \neq 0$
$|J| = a_H q_L - B_L C_H = \ldots$
$= \left(\frac{-1}{r_D} z_D + r_B z_B \left(1 - 2\beta_H r_B z_B\right)\right)\left(-z_D + z \ldots\right)$
$- \left(2\beta_H r_B z_B\right)\left(2\beta_L z_B\right)$
$= z_D^2 / r_D + z_B z_D \left(1 - 2\beta_L\right) \frac{1}{r_D} + \ldots z_B z_D$
$+ r_B z_B^2 (1 - 2\beta_H)(1 - 2\beta_L) - 4 z_B^2$
$\partial_t \Delta_k = -a_L \Delta_k \sum_{w=0}^{k-1} \binom{k}{w} \frac{C_H \langle m_H^w \rangle + \Delta_k}{z_B - z_D} + A_L C_H \sum_{w=0}^{k-1} \binom{k}{w} \frac{a_H \langle m_H^w \rangle - \Delta_k}{z_B - z_D}$
$\partial_t \Delta_k = \frac{a_H \beta_L (A_L - A_H)}{(z_B - z_D)} \sum_{w=0}^{k-1} \binom{k}{w} \langle m_H^w \rangle - \frac{a_L A_H + A_L C_H}{z_B - z_D} \sum_{w=0}^{k-1} \binom{k}{w} \Delta_k$
$\partial_t \langle m_H^k \rangle = \left(z_B r_B - z_D / r_D\right) \frac{C_H \langle m_H^k \rangle + \Delta_k}{z_B - z_D} + \frac{z_B - z_D}{z_B - z_D} \left(a_L \langle m_H^k \rangle - \Delta_k\right)$
$+ \left(r_B \beta_H r_B z_B\right) \sum_{w=0}^{k-1} \binom{k}{w} \frac{C_H \langle m_H^w \rangle + \Delta_k}{z_B - z_D} + z_B \beta_L \sum_{w=0}^{k-1} \binom{k}{w} \frac{a_L \langle m_H^w \rangle - \Delta_k}{z_B - z_D}$
$\frac{r_B z_B - z_D / r_D}{z_B - z_D} = \frac{r_B z_D^{-1} - z_B^{-1} / r_D}{z_D^{-1} - z_B^{-1}} = \frac{r_B \left(h\left(\frac{1}{r_D} - 1\right) + 1\right) - \frac{1}{r_D}\left(h\left(r_B - 1\right) + 1\right)}{\left(h\left(\frac{1}{r_D} - 1\right) + 1\right) - \left(h\left(r_B - 1\right) + 1\right)}$
$= \frac{r_B / r_D h - r_B h + r_B - r_B / r_D h + h / r_D - 1 / r_D}{h / r_D - h + 1 - r_B h + h - 1}$
$= \frac{(h-1)\left(-r_B + 1/r_D\right)}{h / r_D - r_B h} = \frac{-(1+h)}{h} = 1 - 1/h$
$\partial_t \langle m_H^k \rangle = \left(\left(1 - 1/h\right) C_H + a_L\right) \langle m_H^k \rangle + \left(-1/h\right) \Delta_k$
$+ \frac{z_B}{z_B - z_D} \left(a_H \beta_H r_B C_H + 2\beta_H a_L\right) \sum_{w=0}^{k-1} \binom{k}{w} \langle m_H^w \rangle$
$+ \frac{z_B}{z_B - z_D} \left(2\beta_H r_B - 2\beta_L\right) \sum_{w=0}^{k-1} \binom{k}{w} \langle m_H^w \rangle$
$|J| z_D^{-1} z_B^{-1} = \frac{1}{r_D}$
↳ at stationary state, the determinant is always exactly zero (when $r_B = 1/r_D = 1$, this is zero even during transens), so $a_H q_L - B_L C_H = 0$
* the matrix solution is no good
6
10
12
14

BJOL

Calculation Outline: Problem

$$\partial_t \vec{m} = Q\vec{m} + \vec{v}$$

Truncating to the third moment gives:

$$\boxed{|Q| = |J|^3 = \left[Z_D Z_B \left(1/r_D - r_B\right) \partial_t h\right]^3 \to 0}$$

So basic methods are poor in steady-state.

Calculation Outline: Using Iterative Methods

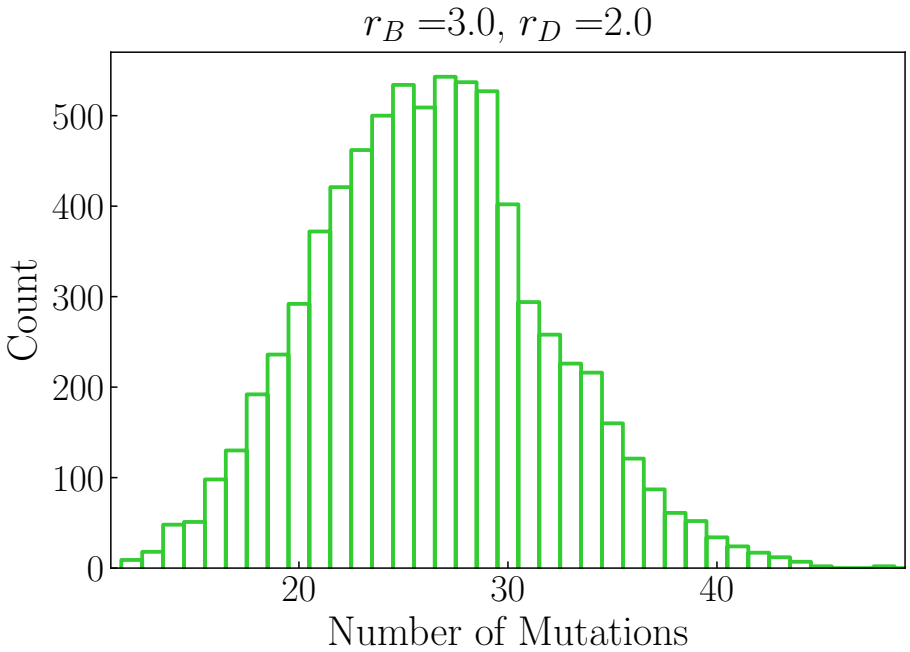
$$\text{Skew} = \frac{\text{Sk}}{\text{Var}^{3/2}}$$

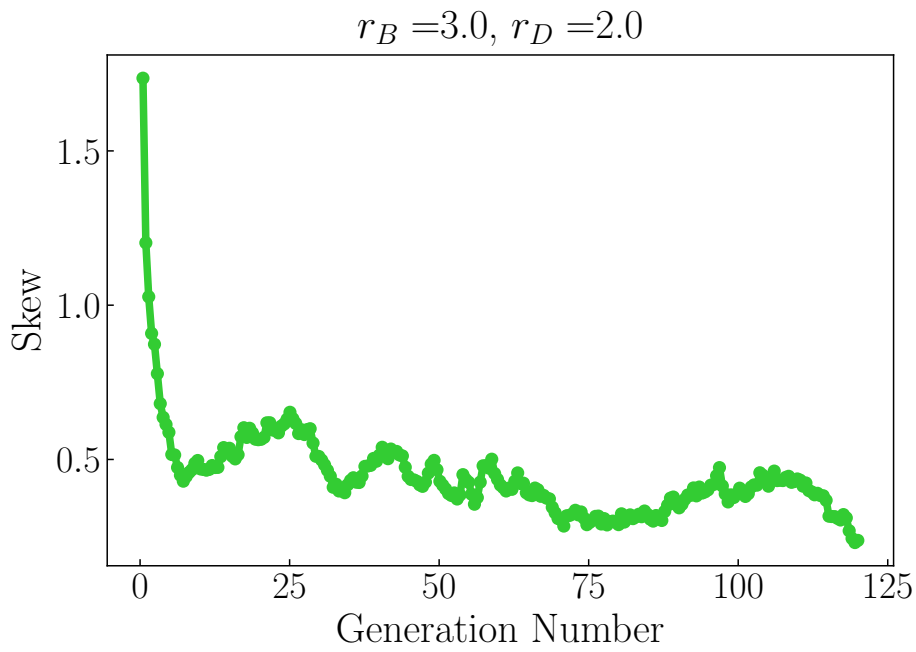
where

$$\text{Sk} = N_3 - 3N_2N_1 + 2N_1^3 = \mathbb{O}(t)$$
$$\text{Var} = N_2 - 2N_1^2 = \mathbb{O}(t)$$

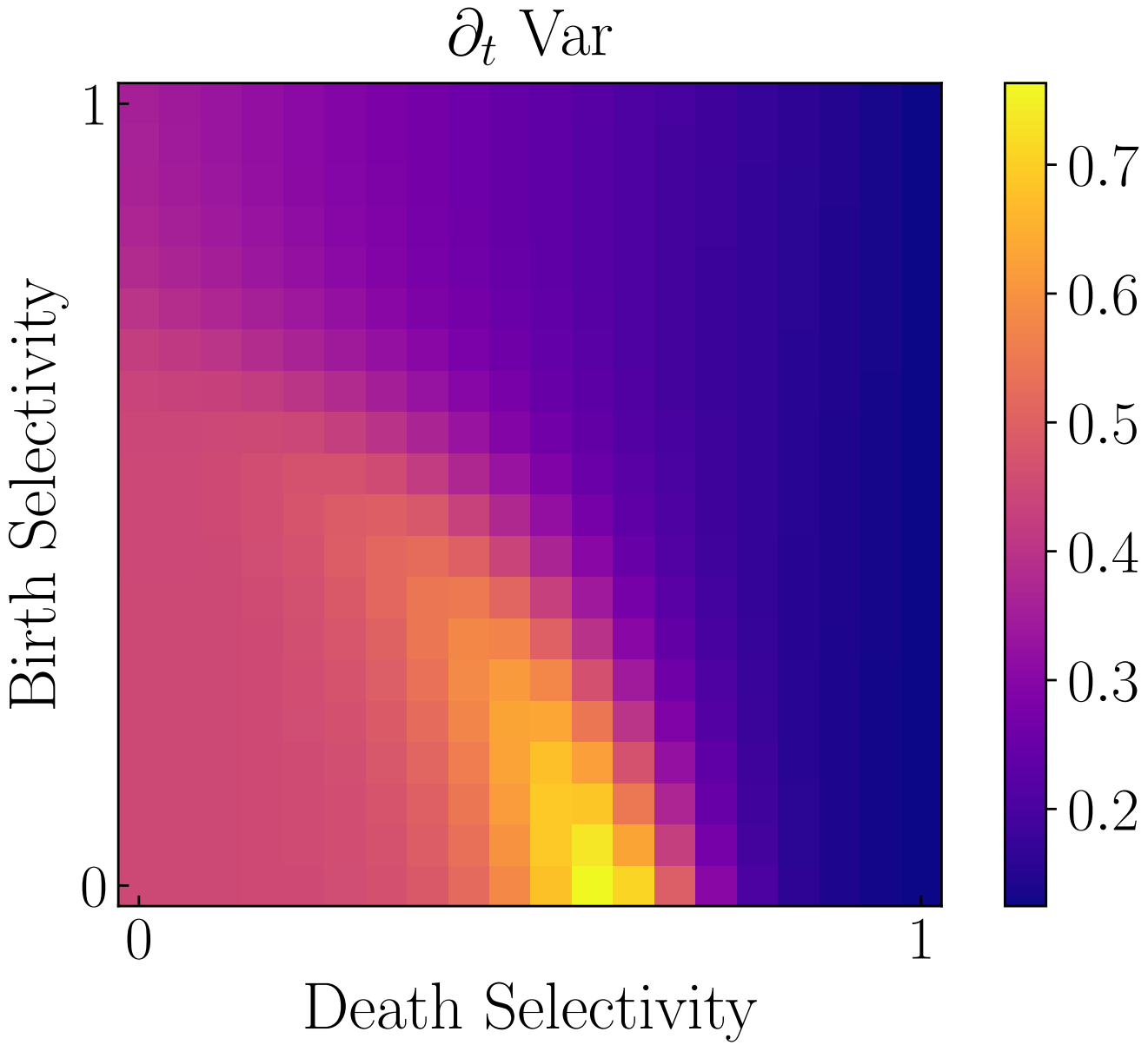
Calculation Outline: The Truth

$$\text{Skew} = \mathbb{O}(t^{-1/2}) \to 0$$





Dynamical Footprint of Selection



Ancestry Hypothesis

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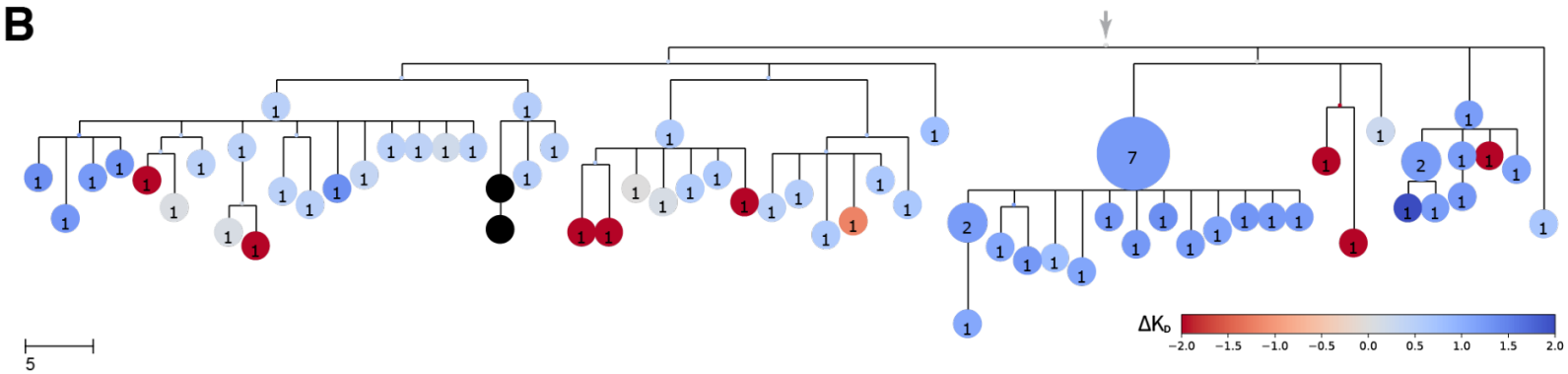


Figure: Tatsuya (2022)

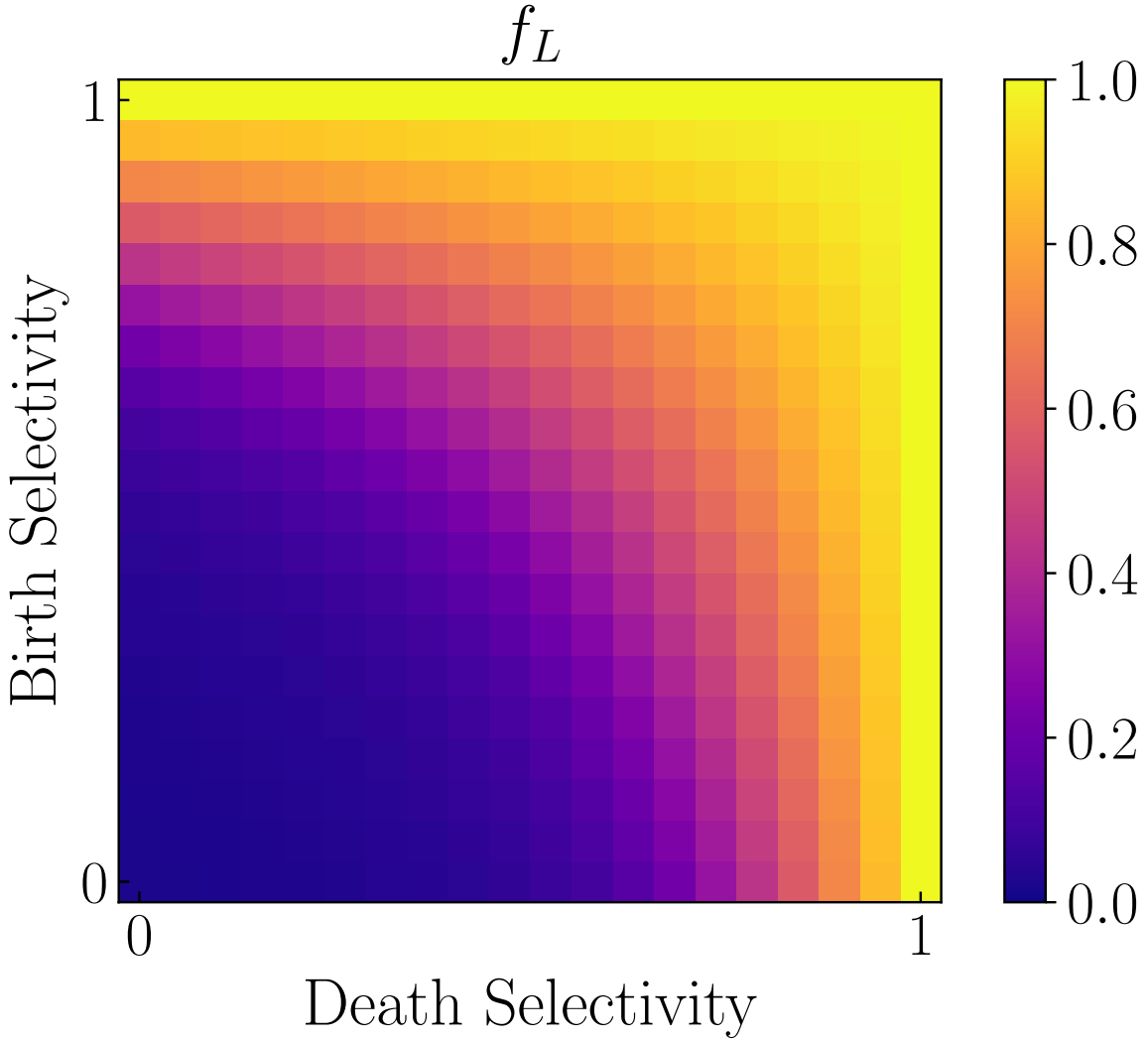
Ancestry Hypothesis

$$f_H = \frac{\text{number of H cells with H ancestors}}{\text{number of H cells}}$$

$$f_L = \frac{\text{number of L cells with H ancestors}}{\text{number of L cells}}$$

Ancestry Hypothesis: False

Notably, f_H and f_L **only** depend on the combined $r_B r_D$.

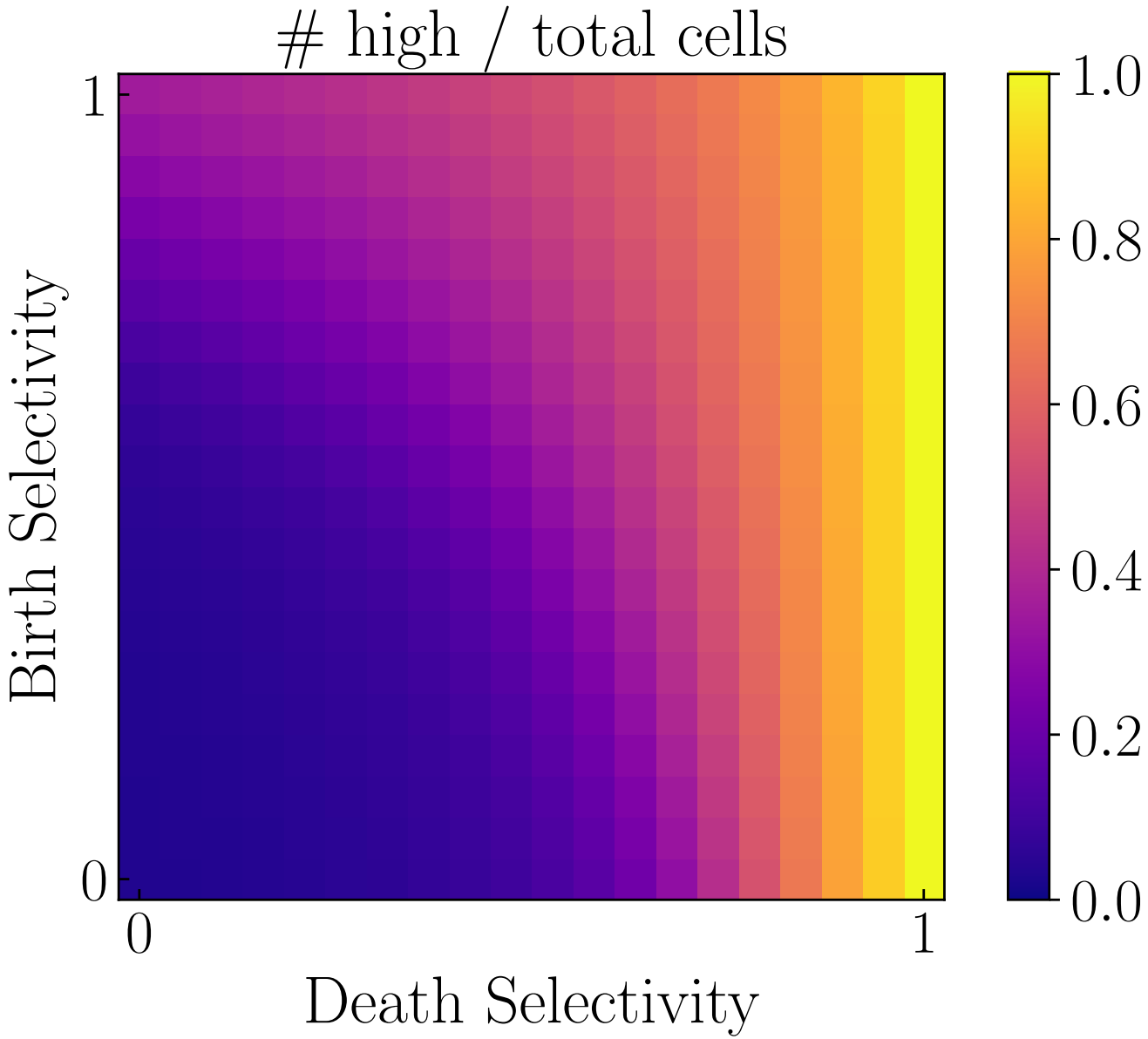


Is there anything static that distiguishes r_D and r_B?

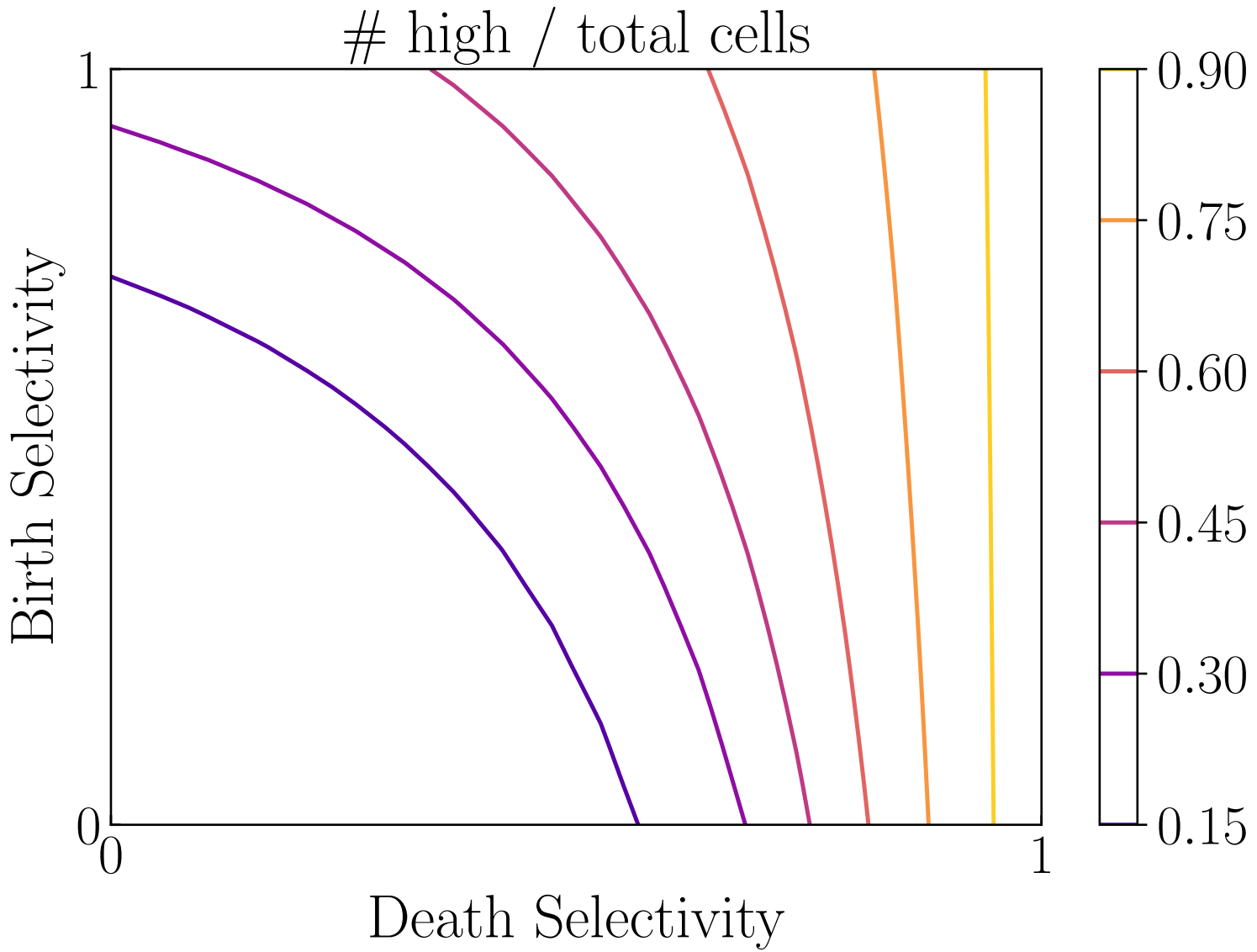
Overall selectivity

$$\boxed{\text{Fraction of cells which are High affinity} = \frac{1}{1 + r_B g(r_B r_D)}}$$

Overall selectivity



Overall selectivity



Static Signals of Selection Scheme?

Mutation Distribution Shape	Nope
Preferential Ancestry	Not Really
Overall Selection Strength	Yep
Extinction and Exit Times	Yep

Thanks to Gabriel Victora, Tatsuya Araki, & Arup Chakraborty for the comments.

Selected References & Image Sources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Talk available at: ottinoloffler.com