Frequency Spirals

Bertrand Ottino-Löffler
Advisor: Steve Strogatz

Cornell University

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This talk

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1. Fireflies!

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2. Heart cells!

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Things synchronize! [1, 2, 3, 4]

1. Fireflies!
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4. Josephson junctions!
5. Applause!
6. Electrical generators!

What these examples have in common.

1. A population of oscillators, with phases $\theta_i \in \mathbb{S}^1$.
2. The oscillators have natural frequencies ω_i, typically drawn from some distribution $g(\omega)$
3. An underlying connectivity graph, showing which oscillators couple to each other.
4. A 2π-periodic coupling function.

Kuramoto

Governing Equation

The *General Kuramoto Model* is given by

$$\dot{\theta}_i = \omega_i + K \sum_{j \in \mathcal{N}(i)} \sin(\theta_j - \theta_i)$$

$K \geq 0$ is the coupling strength, and the sum is over the neighbors $\mathcal{N}(i)$ of oscillator i. K Is the tuning parameter that controls onset of synchronization.

What is synchronization?

Definition

Phase Locking is when $\dot{\theta}_i(t) - \dot{\theta}_j(t) \equiv 0$ for all i and j.

(This is rare).

Definition

A **Frequency-locked cluster** is a set of adjacent oscillators with the same average frequency, as given by
$< \dot{\theta}_i >= \lim_{T\to\infty} \frac{\theta_i(t_0+T)-\theta_i(t_0)}{T}$.

(This is common).

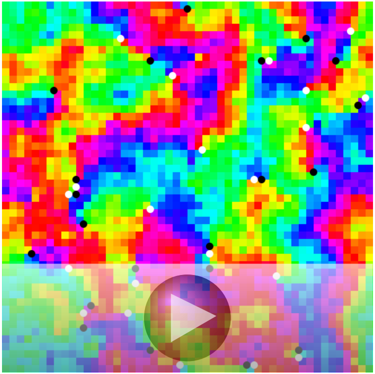
Why use a 2D lattice?

For finite size N, there exists finite $K_E(N)$ where there is a macroscopic frequency-locked cluster.

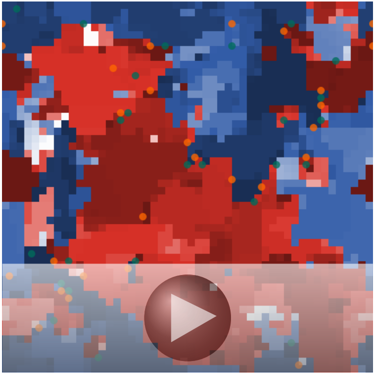
1 Dimension	2 Dimensions	3+ Dimensions
$K_E(N)$ Diverges	$K_E(N) \propto \log(N)$	$K_E(N)$ finite

Table: 2 as critical dimension [5, 6, 7, 8].

2D visualization



(a) Phases



(b) Averages

What were those dots?

Definition

A **Phase Vortex** exists on a 2x2 square of adjacent oscillators if their phases have a lattice curl of $\pm 2\pi$ (The sum of the ordered phase differences mod $(-\pi, \pi]$). This sum can only ever be $+2\pi$,-2π or 0.

Definition

A **Phase Spiral** extends the pattern of a phase vortex to an entire grid, forcing it to have a nonzero winding number around the center.

0	$\pi/4$
$3\pi/2$	$5\pi/6$

Table: A phase vortex

0	$\pi/3$
$\pi/2$	$4\pi/3$

Table: Not a phase vortex

Why vortices?

- Lee et al. observed that the motion of vorticies trace out the boundary of frequency locked clusters, allowing them a partial argument that critical locking strength $K_L \propto \log(N)$ [8].
- Flovik et al. also observed the relevance of "spin waves" (phase spirals) in statistical mechanic's XY model [9].
- Also a phase spiral-esque structure is behind heart fibrulation, a major source of heart failure [10].

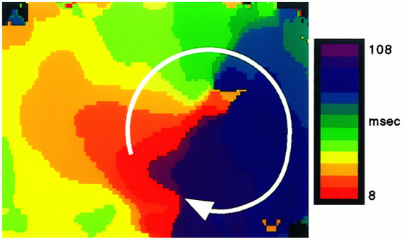
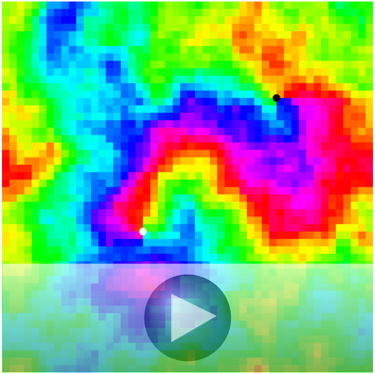
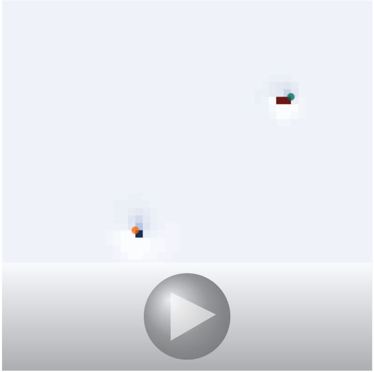


Figure: A phase spiral in a sheep's heart [11].

Close to total entrainment, things get simple.



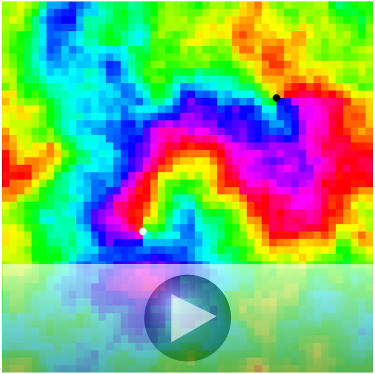
(a) Phases



(b) Averages freqs.

Only two vortices.

Without averaging?



(a) Phases

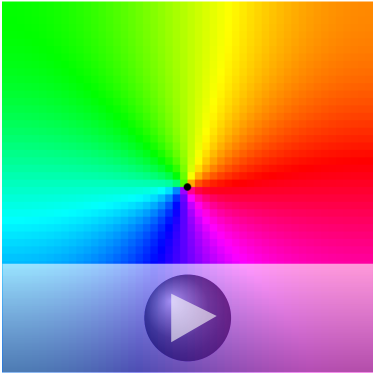
(b) Instantaneous freqs.

New structures are now visible!

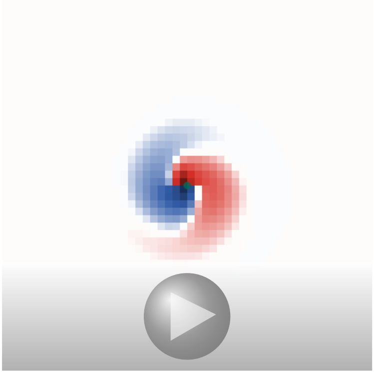
How to make a spiral

1. Start with phase spiral (a system where the phases have a 2π twist around the center)
2. Set all oscillators to have $\omega_j = 0$ execpt for one in the middle, which has a sufficiently nonzero ω.

The Canonical Frequency Spiral

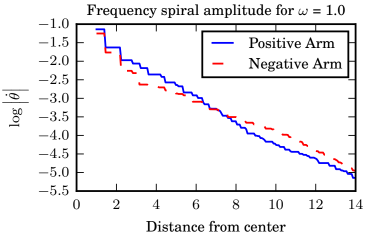


(a) Phases

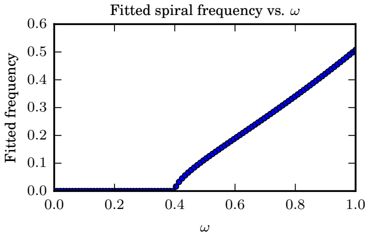


(b) Instantaneous freqs.

Numerical spiral descriptions

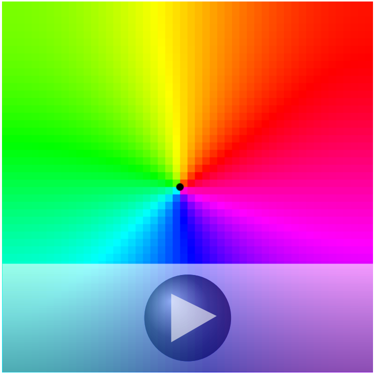


(a) Exponential decays.

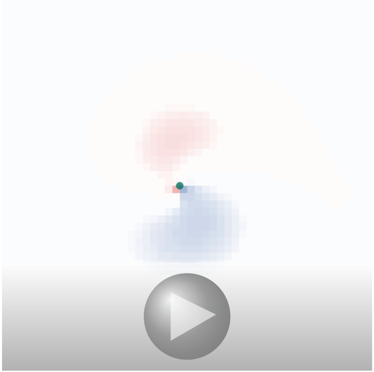


(b) Spiral's frequency

A saddle-node?



(a) Phases



(b) Instantaneous freqs.

Analytic statements are difficult.

Because we are trying to discuss phenomena on an infinite 2D discrete grid, attempts to get anayltic statements about the canonical frequency spiral is difficult.

But maybe we can find a simple workaround?

Go small!

(a) Phases

(b) Instantaneous freqs.

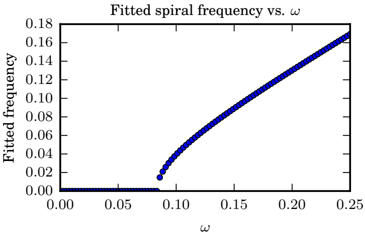
Small enough to analyze

We restrict to exactly five dynamical oscillators with the central oscillator labeled ζ. By some trig substitutions we get:

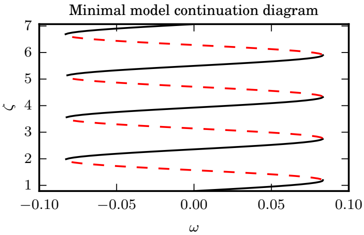
$$
\begin{aligned}
\dot{\theta}_0 &= \sin(\zeta - \theta_0) - \kappa \sin\theta_0, \\
\dot{\theta}_1 &= \sin(\zeta - \theta_1 - \pi/2) - \kappa \sin\theta_1, \\
\dot{\theta}_2 &= \sin(\zeta - \theta_2 - \pi) - \kappa \sin\theta_2, \\
\dot{\theta}_3 &= \sin(\zeta - \theta_3 - 3\pi/2) - \kappa \sin\theta_3, \\
\dot{\zeta} &= \omega - \sin(\zeta - \theta_0) - \sin(\zeta - \theta_1 - \pi/2) \\
&\qquad - \sin(\zeta - \theta_2 - \pi) - \sin(\zeta - \theta_3 - 3\pi/2),
\end{aligned}
$$

where $\kappa = 1 + \sqrt{2}$.

Numerical minimal model descriptions

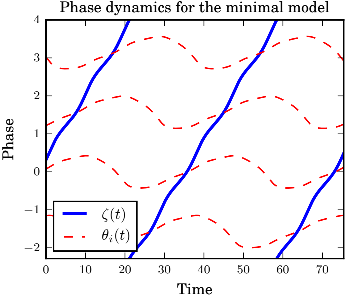


(a) Spiral's frequency

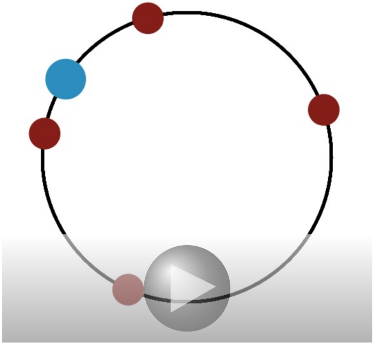


(b) Continuation diagram

Simple phase motion



(a) Simple phase evolution



(b) Phases on circle

Perturbative results

- In addition, this model is simple enough that we can apply Lindstedt's method to get behavior in the limit of extreme ω.
- That is, we rescale time to $\tau = \Omega t/\epsilon$, where $\epsilon := 1/\omega \ll 1$, demand functions be 2π-periodic with respect to τ.
- We then expand Ω, ζ, and θ_0 in terms of ϵ, and equate the coefficients of like powers in ϵ. (e.g., $\Omega = \Omega_0 + \epsilon\Omega_1 + \epsilon^2\Omega_2 + O(\epsilon^3)$)

Skipping over details

Exact expansions

Plug the expansions into the dynamical equations and chug through to get exact perturbative results:

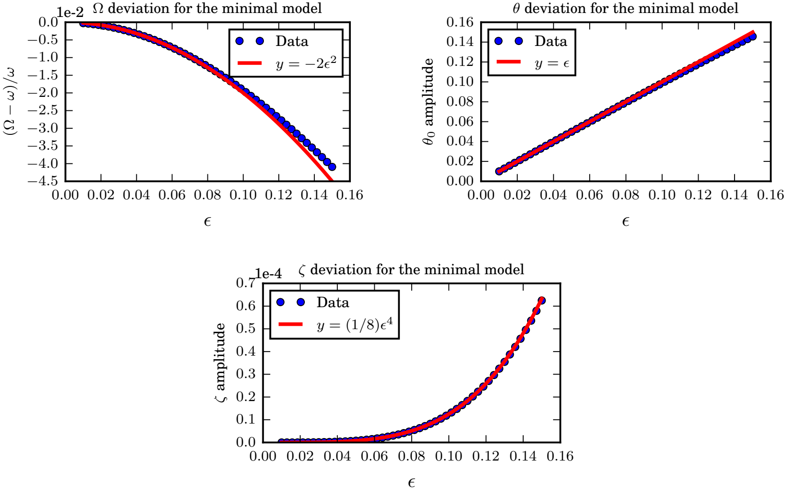
$$\Omega = 1 - 2\epsilon^2 + \epsilon^4 \left(2\kappa^2 - \frac{7}{2} \right) + O(\epsilon^5),$$

$$\zeta(\tau) = \tau + \epsilon^4 \left(\frac{1}{8} \sin 4\tau \right) + O(\epsilon^5),$$

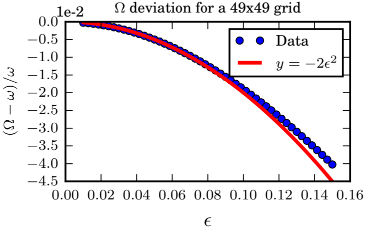
$$\theta_0(\tau) = \epsilon \left(\frac{1}{2\kappa} - \cos \tau \right) + O(\epsilon^2).$$

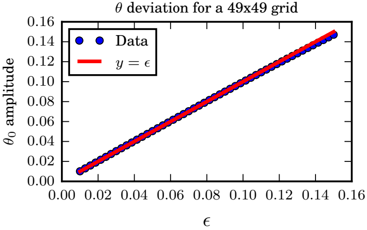
Ω describes the frequency of the spiral, θ_0 describes the amplitude of the spiral one oscillator from the center, and ζ describes the nonuniformity of the spiral.

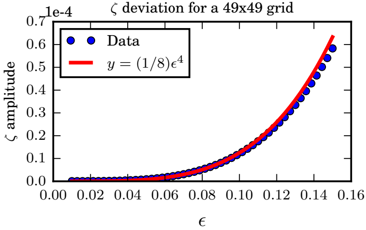
Numerical checks for the minimal model.



Still relevant for a 49x49 grid!







Outro

- We have identified **frequency spirals**, a structure relevant to the onset of synchronization, and have characterized some of its numerical, analytical, and aesthetic aspects.
- However, we would still like to understand how spirals move in space, and interact with one another.
- Using methods from algebraic geometry for the minimal model, what can we learn?
- Characterizing other "single-defector" systems should be fun.

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For slides and further work, see:
`https://people.cam.cornell.edu/~bjo34/`