

Frequency Spirals

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This talk

- 1 General Kuramoto model and features
- 2 Observations on the 2D grid
- 3 Frequency spirals
- 4 Reduction to a minimal model
- 5 Summary and closing

Things synchronize! [1, 2, 3, 4]

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1 Fireflies!

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- 1 Fireflies!
- 2 Heart cells!

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Things synchronize! [1, 2, 3, 4]

- 1 Fireflies!
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- 3 Crickets!
- 4 Josephson junctions!
- 5 Applause!
- 6 Electrical generators!

What these examples have in common.

- 1 A population of oscillators, with phases $\theta_i \in \mathbb{S}^1$.
- 2 The oscillators have natural frequencies ω_i , typically drawn from some distribution $g(\omega)$
- 3 An underlying connectivity graph, showing which oscillators couple to each other.
- 4 A 2π -periodic coupling function.

Governing Equation

The *General Kuramoto Model* is given by

$$\dot{\theta}_i = \omega_i + K \sum_{j \in \mathcal{N}(i)} \sin(\theta_j - \theta_i)$$

$K \geq 0$ is the coupling strength, and the sum is over the neighbors $\mathcal{N}(i)$ of oscillator i . K is the tuning parameter that controls onset of synchronization.

What is synchronization?

Definition

Phase Locking is when $\dot{\theta}_i(t) - \dot{\theta}_j(t) \equiv 0$ for all i and j .

(This is rare).

Definition

A **Frequency-locked cluster** is a set of adjacent oscillators with the same average frequency, as given by

$$\langle \dot{\theta}_i \rangle = \lim_{T \rightarrow \infty} \frac{\theta_i(t_0 + T) - \theta_i(t_0)}{T}.$$

(This is common).

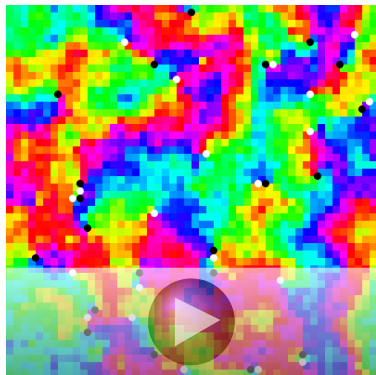
Why use a 2D lattice?

For finite size N , there exists finite $K_E(N)$ where there is a macroscopic frequency-locked cluster.

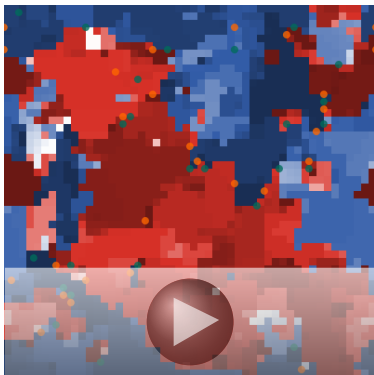
1 Dimension	2 Dimensions	3+ Dimensions
$K_E(N)$ Diverges	$K_E(N) \propto \log(N)$	$K_E(N)$ finite

Table: 2 as critical dimension [5, 6, 7, 8].

2D visualization



(a) Phases



(b) Averages

What were those dots?

Definition

A **Phase Vortex** exists on a 2×2 square of adjacent oscillators if their phases have a lattice curl of $\pm 2\pi$ (The sum of the ordered phase differences mod $(-\pi, \pi]$). This sum can only ever be $+2\pi, -2\pi$ or 0 .

Definition

A **Phase Spiral** extends the pattern of a phase vortex to an entire grid, forcing it to have a nonzero winding number around the center.

0	$\pi/4$
$3\pi/2$	$5\pi/6$

Table: A phase vortex

0	$\pi/3$
$\pi/2$	$4\pi/3$

Table: Not a phase vortex

Why vortices?

- Lee et al. observed that the motion of vortices trace out the boundary of frequency locked clusters, allowing them a partial argument that critical locking strength $K_L \propto \log(N)$ [8].
- Flovik et al. also observed the relevance of "spin waves" (phase spirals) in statistical mechanic's XY model [9].
- Also a phase spiral-esque structure is behind heart fibrillation, a major source of heart failure [10].

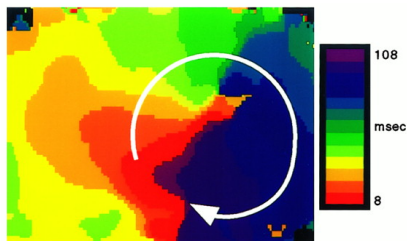
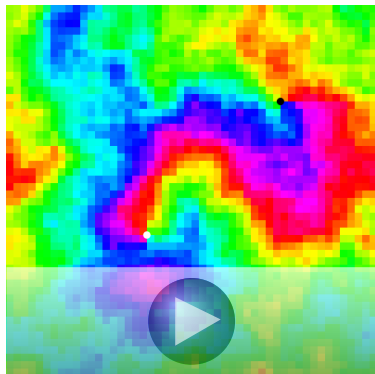
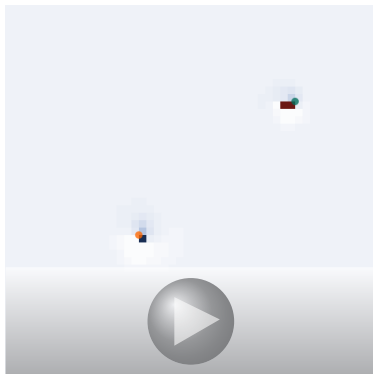


Figure: A phase spiral in a sheep's heart [11].

Close to total entrainment, things get simple.



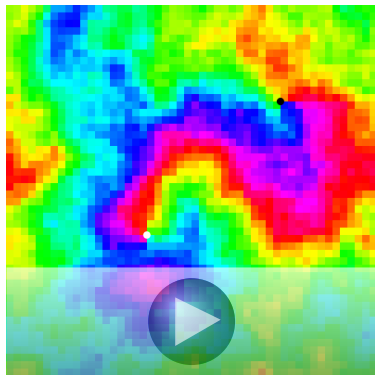
(a) Phases



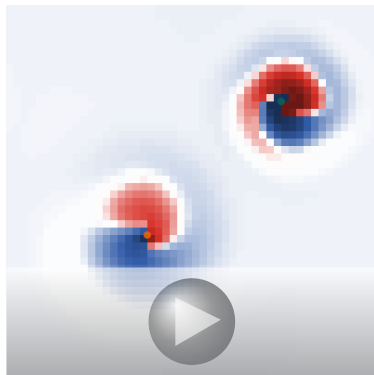
(b) Averages freqs.

Only two vortices.

Without averaging?



(a) Phases



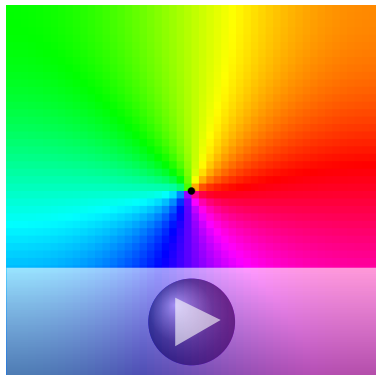
(b) Instantaneous freqs.

New structures are now visible!

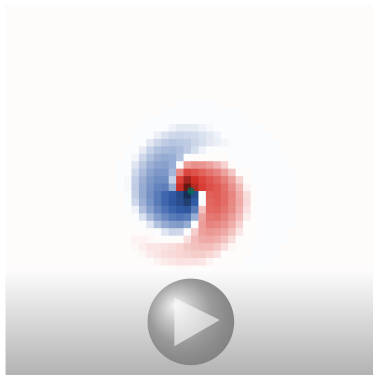
How to make a spiral

- 1 Start with phase spiral (a system where the phases have a 2π twist around the center)
- 2 Set all oscillators to have $\omega_j = 0$ except for one in the middle, which has a sufficiently nonzero ω .

The Canonical Frequency Spiral

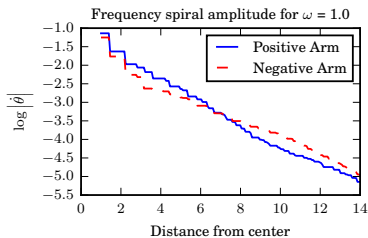


(a) Phases

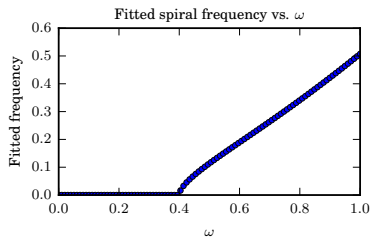


(b) Instantaneous freqs.

Numerical spiral descriptions

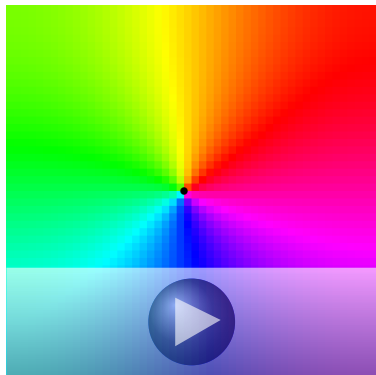


(a) Exponential decays.

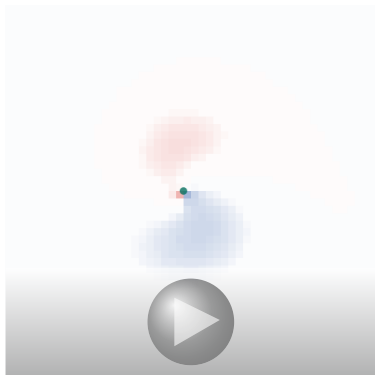


(b) Spiral's frequency

A saddle-node?



(a) Phases



(b) Instantaneous freqs.

Analytic statements are difficult.

Because we are trying to discuss phenomena on an infinite 2D discrete grid, attempts to get analytic statements about the canonical frequency spiral is difficult.

But maybe we can find a simple workaround?

Go small!



(a) Phases



(b) Instantaneous freqs.

Small enough to analyze

We restrict to exactly five dynamical oscillators with the central oscillator labeled ζ . By some trig substitutions we get:

$$\dot{\theta}_0 = \sin(\zeta - \theta_0) - \kappa \sin \theta_0,$$

$$\dot{\theta}_1 = \sin(\zeta - \theta_1 - \pi/2) - \kappa \sin \theta_1,$$

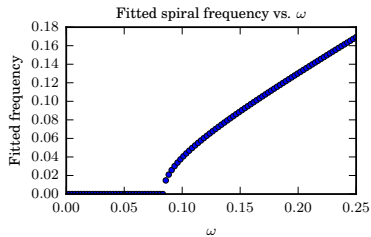
$$\dot{\theta}_2 = \sin(\zeta - \theta_2 - \pi) - \kappa \sin \theta_2,$$

$$\dot{\theta}_3 = \sin(\zeta - \theta_3 - 3\pi/2) - \kappa \sin \theta_3,$$

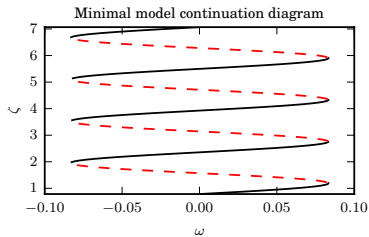
$$\begin{aligned} \dot{\zeta} = \omega &- \sin(\zeta - \theta_0) - \sin(\zeta - \theta_1 - \pi/2) \\ &- \sin(\zeta - \theta_2 - \pi) - \sin(\zeta - \theta_3 - 3\pi/2), \end{aligned}$$

where $\kappa = 1 + \sqrt{2}$.

Numerical minimal model descriptions

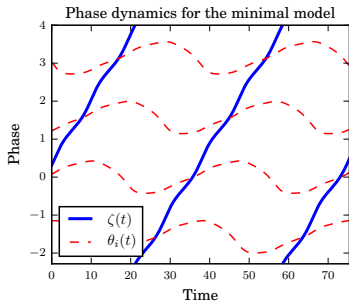


(a) Spiral's frequency

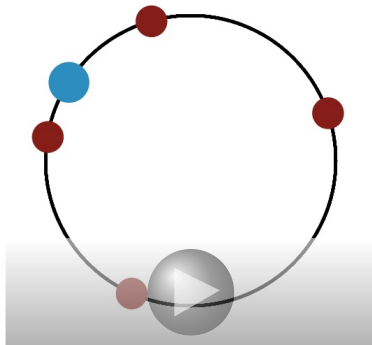


(b) Continuation diagram

Simple phase motion



(a) Simple phase evolution



(b) Phases on circle

Perturbative results

- In addition, this model is simple enough that we can apply Lindstedt's method to get behavior in the limit of extreme ω .
- That is, we rescale time to $\tau = \Omega t / \epsilon$, where $\epsilon := 1/\omega \ll 1$, demand functions be 2π -periodic with respect to τ .
- We then expand Ω , ζ , and θ_0 in terms of ϵ , and equate the coefficients of like powers in ϵ . (e.g.,
$$\Omega = \Omega_0 + \epsilon\Omega_1 + \epsilon^2\Omega_2 + O(\epsilon^3)$$
)

Skipping over details

Lindstead.nb

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$$\begin{aligned} & \text{Omg1 Phi2'[t] - Omg0 Phi3'[t] = 0 \& \& k \left(-\frac{1}{6} \text{Cos[Phi0[t]] Phi1[t]^3 - Cos[Phi0[t]] Phi3[t] - Phi1[t] Phi2[t] Sin[Phi0[t]] \right) - \frac{1}{6} \text{Cos[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t])^3 -} \\ & \text{Sin[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t]) (-Phi2[t] - Zet2[t]) - Cos[Phi0[t] - Zet0[t]] (-Phi3[t] - Zet3[t]) + Omg4 Phi0'[t] + Omg3 Phi1'[t] + Omg2 Phi2'[t] + Omg1 Phi3'[t] - Omg0 Phi4'[t] =} \\ & \text{0 \& \& k \left(-\frac{1}{2} \text{Cos[Phi0[t]] Phi1[t]^2 Phi2[t] + Cos[Phi0[t]] Phi4[t] + \frac{1}{24} Phi1[t]^4 Sin[Phi0[t]] - \frac{1}{2} (Phi2[t]^2 + 2 Phi1[t] Phi3[t]) Sin[Phi0[t]] \right) + \frac{1}{24} Sin[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t])^4 -} \\ & \frac{1}{2} \text{Cos[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t])^2 (-Phi2[t] + Zet2[t]) - \frac{1}{2} Sin[Phi0[t] - Zet0[t]] (-Phi2[t] + Zet2[t])^2 - 2 (-Phi1[t] + Zet1[t]) (-Phi3[t] + Zet3[t]) -} \\ & \text{Cos[Phi0[t] - Zet0[t]] (-Phi4[t] - Zet4[t]) - Omg5 Phi0'[t] - Omg4 Phi1'[t] - Omg3 Phi2'[t] - Omg2 Phi3'[t] - Omg1 Phi4'[t] - Omg0 Phi5'[t] = 0 \& \& } \\ & \text{k \left(\frac{1}{120} \text{Cos[Phi0[t]] Phi1[t]^3 - \frac{1}{6} Cos[Phi0[t]] (2 Phi1[t] Phi2[t]^2 + Phi1[t]^2 Phi3[t] + Phi1[t] (Phi2[t]^2 + 2 Phi1[t] Phi3[t])) +} \right.} \\ & \left. \text{Cos[Phi0[t]] Phi5[t] + \frac{1}{6} Phi1[t]^2 Phi2[t] Sin[Phi0[t]] - \frac{1}{2} (2 Phi2[t] Phi3[t] + 2 Phi1[t] Phi4[t]) Sin[Phi0[t]] \right) -} \\ & \frac{1}{120} \text{Cos[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t])^3 - \frac{1}{6} Sin[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t])^2 (-Phi2[t] + Zet2[t]) - \frac{1}{6} \text{Cos[Phi0[t] - Zet0[t]]} \\ & \left(2 (-Phi1[t] + Zet1[t]) (-Phi2[t] + Zet2[t])^2 + (-Phi3[t] + Zet3[t]) + (-Phi1[t] + Zet1[t]) ((-Phi2[t] + Zet2[t])^2 + 2 (-Phi1[t] + Zet1[t]) (-Phi3[t] + Zet3[t])) \right) +} \\ & \frac{1}{2} \text{Sin[Phi0[t] - Zet0[t]] (-2 (-Phi2[t] + Zet2[t]) (-Phi3[t] + Zet3[t]) - 2 (-Phi1[t] + Zet1[t]) (-Phi4[t] + Zet4[t])) - Cos[Phi0[t] - Zet0[t]] (-Phi5[t] + Zet5[t]) +} \\ & \text{Omg6 Phi0'[t] - Omg5 Phi1'[t] - Omg4 Phi2'[t] - Omg3 Phi3'[t] - Omg2 Phi4'[t] - Omg1 Phi5'[t] - Omg0 Phi6'[t] = 0} \end{aligned}$$

DetEqs = LogicalExpand[EQ2]

$$\begin{aligned} & \text{1 - Omg0 Zet0'[t] = 0 \& \& Cos[Phi0[-\frac{\pi}{2} - t] - Zet0[t]] - Cos[Phi0[\frac{\pi}{2} - t] - Zet0[t]] - Sin[Phi0[t] - Zet0[t]] - Sin[Phi0[-\pi - t] - Zet0[t]] - Omg1 Zet0'[t] - Omg0 Zet1'[t] = 0 \& \& } \\ & \text{-Cos[Phi0[-\pi + t] - Zet0[t]] (Phi1[-\pi + t] - Zet1[t]) - Sin[Phi0[-\frac{\pi}{2} + t] - Zet0[t]] (Phi1[-\frac{\pi}{2} + t] - Zet1[t]) -} \\ & \text{Cos[Phi0[t] - Zet0[t]] (-Phi4[t] - Zet4[t]) - Sin[Phi0[\frac{\pi}{2} + t] - Zet0[t]] (-Phi1[\frac{\pi}{2} + t] - Zet1[t]) - Omg2 Zet0'[t] - Omg1 Zet1'[t] - Omg0 Zet2'[t] = 0 \& \& } \\ & \text{-\frac{1}{2} Sin[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t])^2 - \frac{1}{2} Sin[Phi0[-\pi - t] - Zet0[t]] (-Phi1[-\pi - t] - Zet1[t])^2 - \frac{1}{2} \text{Cos[Phi0[-\frac{\pi}{2} - t] - Zet0[t]] (-Phi1[-\frac{\pi}{2} - t] - Zet1[t])^2 -} \\ & \frac{1}{2} \text{Cos[Phi0[\frac{\pi}{2} + t] - Zet0[t]] (-Phi1[\frac{\pi}{2} + t] + Zet1[t])^2 - Cos[Phi0[-\pi + t] - Zet0[t]] (Phi2[-\pi + t] - Zet2[t]) - Sin[Phi0[-\frac{\pi}{2} + t] - Zet0[t]] (Phi2[-\frac{\pi}{2} + t] - Zet2[t]) -} \\ & \text{Cos[Phi0[t] - Zet0[t]] (-Phi2[t] - Zet2[t]) - Sin[Phi0[\frac{\pi}{2} + t] - Zet0[t]] (-Phi2[\frac{\pi}{2} + t] - Zet2[t]) - Omg3 Zet0'[t] - Omg2 Zet1'[t] - Omg1 Zet2'[t] - Omg0 Zet3'[t] = 0 \& \& } \\ & \frac{1}{6} \text{Cos[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t])^3 - \frac{1}{6} \text{Cos[Phi0[-\pi - t] - Zet0[t]] (-Phi1[-\pi - t] - Zet1[t])^3 - \frac{1}{6} Sin[Phi0[-\frac{\pi}{2} - t] - Zet0[t]] (-Phi1[-\frac{\pi}{2} - t] - Zet1[t])^3 -} \\ & \frac{1}{6} \text{Sin[Phi0[\frac{\pi}{2} + t] - Zet0[t]] (-Phi1[\frac{\pi}{2} + t] + Zet1[t])^3 - Sin[Phi0[t] - Zet0[t]] (-Phi1[t] + Zet1[t]) (-Phi2[t] - Zet2[t]) +} \end{aligned}$$

Exact expansions

Plug the expansions into the dynamical equations and chug through to get exact perturbative results:

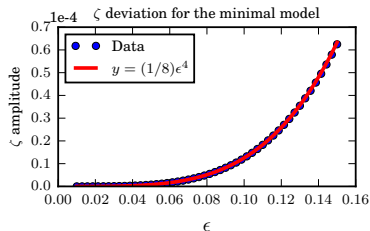
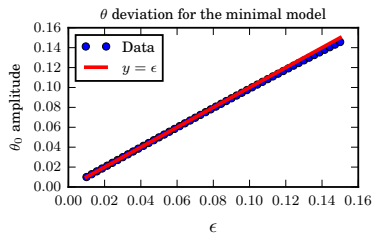
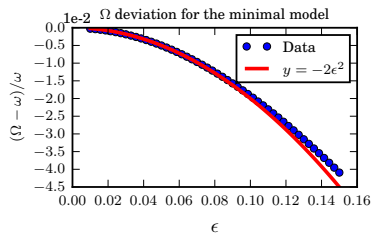
$$\Omega = 1 - 2\epsilon^2 + \epsilon^4 \left(2\kappa^2 - \frac{7}{2} \right) + O(\epsilon^5),$$

$$\zeta(\tau) = \tau + \epsilon^4 \left(\frac{1}{8} \sin 4\tau \right) + O(\epsilon^5),$$

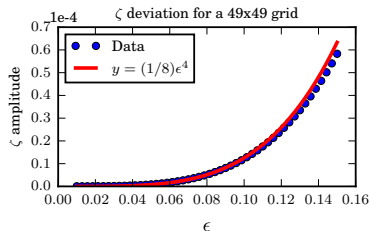
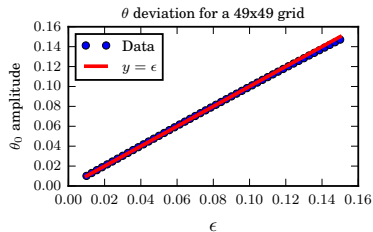
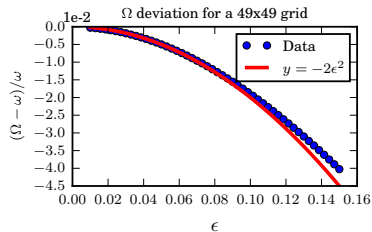
$$\theta_0(\tau) = \epsilon \left(\frac{1}{2\kappa} - \cos \tau \right) + O(\epsilon^2).$$

Ω describes the frequency of the spiral, θ_0 describes the amplitude of the spiral one oscillator from the center, and ζ describes the nonuniformity of the spiral.

Numerical checks for the minimal model.



Still relevant for a 49x49 grid!



- We have identified **frequency spirals**, a structure relevant to the onset of synchronization, and have characterized some of its numerical, analytical, and aesthetic aspects.
- However, we would still like to understand how spirals move in space, and interact with one another.
- Using methods from algebraic geometry for the minimal model, what can we learn?
- Characterizing other "single-defector" systems should be fun.

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For slides and further work, see:
<https://people.cam.cornell.edu/~bjo34/>