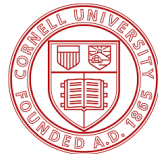


# A volcano transition in a solvable model of oscillator glass

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04/10/19



# Outline

- 1 Lead in to glass
- 2 Daido's volcano
- 3 An interaction model
- 4 Analytic calculation of bifurcation
- 5 The numerics
- 6 Glass verification
- 7 Conclusion

# Oscillators?

# Oscillator ingredients

- Each oscillator has a **phase**  $\theta$ .
- Each oscillator has a **natural frequency**  $\omega$ .
- Oscillators **couple** to one another.
- There is a tunable **coupling strength**  $J$  between oscillators.

# Main dynamics

$$\dot{\theta}_j = \omega_j + \sum_{k=1}^N J_{jk} \sin(\theta_k - \theta_j) \text{ for } j = 1, \dots, N. \quad (1)$$

Background: glass?

# Background: what is a glass?

- Slow.
- Messy.
- Complicated.
- Frustrated.

# Background: what is a glass?

- Slow, non-exponential/algebraic decay of chosen order parameter to steady states.
- Substantial lack of long-range order or symmetry.
- Significant redundancy in the set of possible ground microstates that all carry similar macroscopic behavior.
- Frustrated.



# What does frustration look like?

Phase Video!

Caption

# Can oscillators make a “true” glass?

## Self-averaging of an order parameter in randomly coupled limit-cycle oscillators

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(Received 11 August 1999)

In our recent paper [Phys. Rev. E **58**, 1789 (1998)] we found notable deviations from a power-law decay for the “magnetization” of a system of coupled phase oscillators with random interactions claimed by Daido in Phys. Rev. Lett. **68**, 1072 (1992). For another long-time property, the Lyapunov exponent, we found that his numerical procedure showed strong time discretization effects and we suspected a similar effect for the algebraic decay. In the Comment to our paper [preceding paper, Phys. Rev. E **61**, 2145 (2000)] Daido made clear that the power law behavior was only claimed for the sample averaged magnetization  $[Z]$  and he presented new, more accurate numerical results which provide evidence for a power-law decay of this quantity. Our results, however, were obtained for  $Z$  itself and not for  $[Z]$ . In addition, we have taken the intrinsic oscillator frequencies as Gaussian random variables, while Daido in his new and apparently also in his earlier simulations used a deterministic approximation to the Gaussian distribution. Due to the differences in the observed quantity and the model assumptions our and Daido’s results may be compatible.

PACS number(s): 05.45.-a

The investigation of interacting biological oscillators was introduced by Winfree [1] several decades ago. Building on Winfree’s idea of a phase description, Kuramoto introduced a simple model of interacting phase oscillators [2–5], which could be solved analytically in the limit of  $N \rightarrow \infty$  coupled oscillators. The dynamics of the phases of the oscillators are described by coupled first order differential equations:

noting the interaction strength. The frequencies  $\omega_j$  are distributed according to a Gaussian distribution function  $f(\omega) = \exp(-\omega^2/2)/\sqrt{2\pi}$ .

A continuous distribution function  $f(\omega)$  is only well-defined for  $N \rightarrow \infty$  or for random variables  $\omega_j$ . Although not stated in [7], in his numerical calculations (with finite  $N$ ) Daido, in contrast to us [10], used nonrandom frequencies [see Eq. (3) in his comment [11]].

Daido investigated the decay of  $Z(t)$  from the initial con-

## Algebraic relaxation of an order parameter in randomly coupled limit-cycle oscillators

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(Received 28 December 1998)

In their recent paper [Phys. Rev. E **58**, 1789 (1998)], Stiller and Radons (SR) study, following our earlier work [Phys. Rev. Lett. **68**, 1073 (1992)], the behavior of globally and randomly coupled phase oscillators with distributed intrinsic frequencies. They claim that their simulation results do not confirm the power-law behavior of an order parameter found numerically by the author, attributing its cause to the poor precision of the author's integration scheme. Here demonstrated is that the power law survives even for a scheme better than SR's, provided that finite-size effects are properly taken into account, as was done in our previous work.

PACS number(s): 05.45.-a, 87.10.+c, 02.50.-r, 05.40.-a

The behavior of large populations of coupled nonlinear oscillators is now one of the central subjects in nonlinear dynamics [1-3]. Although it is usually studied for the case of nonrandom interactions, the architecture of coupling in any kind of real coupled-oscillator systems should involve more or less quenched disorder. If such disorder is weak enough, then it will not cause any significant change in the system's behavior. However, one may expect the emergence of qualitatively new features when randomness in coupling is not decoratively feeble. For example, many biological and physiological oscillator systems, including the brain, might be examples of such a case [1,4]. It is therefore an important and interesting subject to examine the behavior of randomly coupled oscillators. From this point of view, the author started investigations into pools of randomly coupled limit-cycle oscillators more than a decade ago [5] and later proposed a model of "oscillator glass" [6],

when  $J$  exceeds  $J_c$ , a threshold value, where  $[Z(t)]$  stands for an average of  $Z(t)$  over a number of different realizations of  $J_{ij}$ , which average will hereafter be referred to as a "sample-average" following Refs. [6,11] and the number of realizations used will be denoted by  $N_s$  [12]. In that work, numerical integration was performed with the Euler scheme of time step  $\Delta t = 2\pi \times 0.01$  for  $N = 500, 1000, 2000$  and  $N_s = 10$ . For the larger values of  $N$ ,  $J_c$  is near 6, and the exponent  $\alpha$  depends on  $J$  [see Fig. 5(b) of Ref. [6]].

Recently, however, Stiller and Radons (SR) [13] have reported that their results are essentially different: they find exponential decay for  $J < 24.5$  and algebraic decay only for  $J = 24.5$ , leaving the region  $J > 24.5$  unsettled because of the complex behavior of the order parameter therein. Their integration scheme is the Heun method with  $\Delta t = 2\pi \times 10^{-3}$ , which is expected to have better accuracy than the author's [6]. From this fact, they suggest that the slow relaxation found by the author is not correct, being a discretization effect due to the low order integration scheme. The main

**Dynamics of nonlinear oscillators with random interactions**

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(Received 3 October 1997)

We develop a mean field theory for a system of coupled oscillators with random interactions with variable symmetry. Numerical simulations of the resulting one-dimensional dynamics are in accordance with simulations of the  $N$ -oscillator dynamics. We find a transition in dependence on interaction strength  $J$  and symmetry parameter  $\eta$  from a dynamically disordered phase to a phase with static disorder, where all oscillators are frozen in random positions. This transition between the "paramagnetic" phase and the spin glass phase appears to be of first order and is dynamically characterized by chaos (positive Lyapunov exponents) in the former case and regular motion (vanishing Lyapunov exponents) in the latter case. The Lyapunov spectrum shows an interesting symmetry for antisymmetric interaction ( $\eta = -1$ ). [S1063-651X(98)14608-1]

PACS number(s): 05.45.+b, 05.70.Fh, 64.60.Ht, 64.60.Cn

**I. INTRODUCTION**

Oscillations and interacting oscillating systems are omnipresent in nature as well as in technical systems. Therefore, systems of coupled oscillators have received much interest in the last years. Synchronization and desynchronization were investigated for populations of fireflies [1,2], pacemaker cells of the heart, and pulsating lasers [3,39]. Oscillations in the nervous system [5], which control periodical processes as running, breathing, and chewing, received particular interest

tion"  $m = (1/N) \sum_{j=1}^N \exp(i\phi_j)$  according to a power law in time. Our simulations with identical system size, smaller time discretization, and a numerical procedure of higher order do not confirm this result. We find a power law only for a critical interaction strength  $J_c$ ; for  $J > J_c$  and  $J < J_c$ , we do find systematic deviations from a power law. At  $J_c$ , however, the system shows a discontinuous transition from a dynamically disordered state to a spin glass state with frozen disorder, in contrast to the case of uniform and Van Heman type [26] interactions, where the transition is con-

## Quasientrainment and Slow Relaxation in a Population of Oscillators with Random and Frustrated Interactions

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(Received 31 January 1991; revised manuscript received 22 October 1991)

It is numerically shown that there may be a new type of ordered state (in some sense glassy) in far-from-equilibrium systems which can be identified with a large population of coupled limit-cycle oscillators, provided couplings are not only random but also frustrated. It is characterized by quasientrainment and algebraic relaxation.

PACS numbers: 87.10.+e, 02.50.+s, 05.40.+j, 05.70.Fh

Large assemblies of coupled limit-cycle oscillators play an important role in many fields of science. Their most remarkable feature is that they exhibit macroscopic mutual entrainment for coupling strength greater than a certain threshold. The resulting coherent oscillations model a variety of rhythmic behaviors observed in diverse far-from-equilibrium systems, such as biological clocks, many physiological organisms, chemical reactors, and so on [1-3]. Quite a few investigations have been carried out analytically as well as numerically for a number of model systems in order to elucidate the nature of such a transition [2].

A common feature of the models used in those studies

of coupled oscillators [4,7]. Suppose a pair of oscillators evolve as  $\dot{\theta}_1 = \bar{J} \sin 2\pi(\theta_2 - \theta_1 + \alpha_{21})$ ,  $\dot{\theta}_2 = \bar{J} \sin 2\pi(\theta_1 - \theta_2 + \alpha_{12})$ , whose phase difference asymptotically becomes  $\theta_1 - \theta_2 = \beta_{12} + \sigma_{12}$ , where  $\beta_{12} \equiv (a_{21} - a_{12})/2$  and  $\sigma_{12} = 0$  or  $\frac{1}{2}$  depending on the value of  $a_{12} + a_{21}$ . Then, imagine three oscillators, every pair of which has a similar coupling as above. It is easy to see that unless  $\sum_{\text{pair}} (\beta_{ij} + \sigma_{ij}) = 0 \pmod{1}$ , the phase difference favored by the coupling cannot come true for *all* of the pairs, leading to a competition among interactions, or frustration. Extension is straightforward to the case of more than three oscillators.

Frustration may be a fairly common feature of real

# Main dynamics: Daido's choices

$$\dot{\theta}_j = \omega_j + \sum_{k=1}^N J_{jk} \sin(\theta_k - \theta_j) \text{ for } j = 1, \dots, N.$$

- Normally distributed natural frequencies.
- Symmetric coupling matrix ( $J_{jk} = J_{kj}$ ), with entries distributed as  $\text{Normal}(0, 2\pi J/\sqrt{N})$ .

# Daido's dynamics: $J < 8$

Phase Video!

Caption



# Daido's dynamics: $J > 8$

Phase Video!

Caption

# Local Fields

We define the **local fields** to be

$$P_j = r_j e^{i\phi_j} := \sum_{k=1}^N J_{jk} e^{i\theta_k},$$

for  $j = 1, \dots, N$ . Equation (1) then becomes

$$\dot{\theta}_j = \omega_j + r_j \sin(\phi_j - \theta_j).$$

## Daido (1992): Local field trajectories

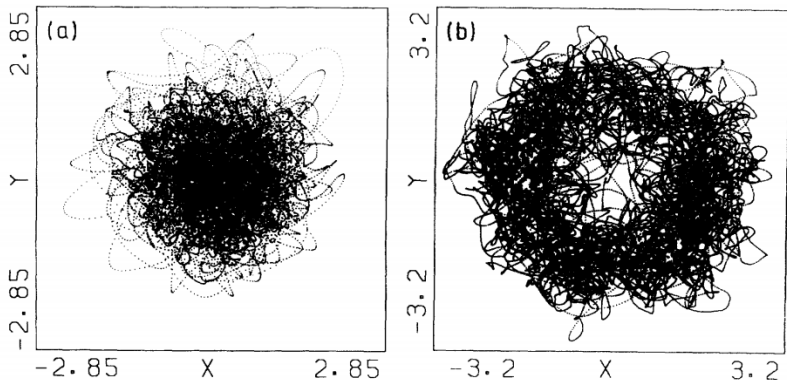


FIG. 1. Phase portraits of a local field ( $N = 500$ ): (a)  $J = 4$  (20000 points); (b)  $J = 14$  (50000 points).

# Daido's dynamics: $J < 8$

Field Video!

Caption

Phase Video!

Caption

# Daido's dynamics: $J > 8$

Field Video!

Caption

Phase Video!

Caption

## Daido (1992): a volcano transition

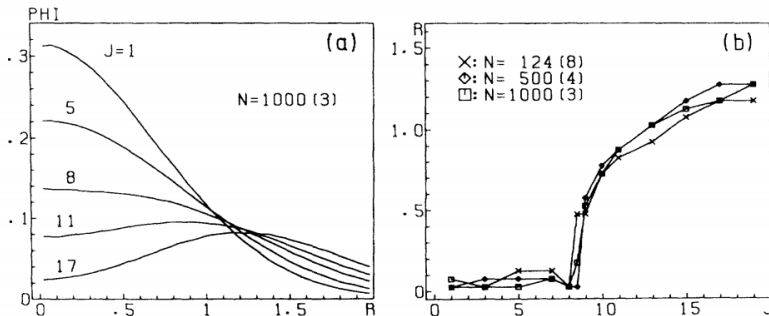


FIG. 2. Behavior of the distribution of a local field averaged over all  $j$ : (a)  $\text{PHI}(R) \equiv P(x, y)$  vs  $R = (x^2 + y^2)^{1/2}$ , where  $P$  is averaged over three samples of  $J_{ij}$  as is expressed by (3) in the figure (this convention will be used hereafter); (b) the peak point of  $\text{PHI}(R)$  vs  $J$ .

# Daido (1992): a volcano transition

## Claim

*There is some critical coupling scale  $J_c$  that corresponds to the onset of a volcano transition. Moreover, this corresponds to a transition into a glassy state.*

# Main dynamics: our choices



# Main dynamics: our choices

Instead of normally distributed natural frequencies, we use the Cauchy distribution

$$g(\omega) = \frac{1}{\pi(1 + \omega^2)}.$$

# Main dynamics: our choices

Instead of normally distributed couplings, we make

$$J_{jk} = \frac{J}{N} \sum_{m=1}^K (-1)^m u_m^{(j)} u_m^{(k)}, \quad (2)$$

where each  $u$  is iid, with equal probability of being  $\pm 1$ .

## The coupling matrix: special features

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- $J$  controls the spread.
- $K$  controls the rank, since it decides the number of outer products making  $J_{jk}$ .
- Each oscillator  $j$  has a vector  $(u_1^{(j)}, \dots, u_K^{(j)})$ , and the coupling between oscillators  $j$  and  $k$  depends on the number of places their vectors agree.

# The coupling matrix: extra special features

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- If  $K = N$  and  $N \rightarrow \infty$ , then the off-diagonal entries go to  $\text{Normal}(0, J/\sqrt{N})$ . (**Has relevant limit**)
- Entries of  $J_{jk}$  are **independent**. (Are they?)

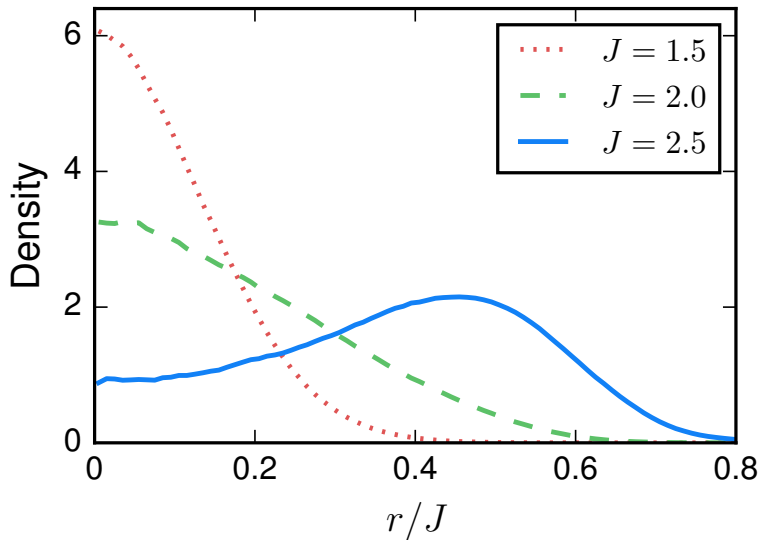
# Local Fields (reminder)

We define the local fields to be

$$P_j = r_j e^{i\phi_j} := \sum_{k=1}^N J_{jk} e^{i\theta_k},$$

for  $j = 1, \dots, N$ . Equation (1) then becomes

$$\dot{\theta}_j = \omega_j + r_j \sin(\phi_j - \theta_j).$$



# Local Fields for $J = 1$

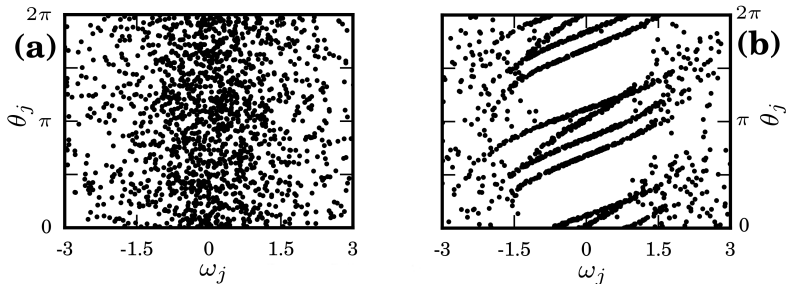
**Figure:** [Field Video!](#)

[Full Caption](#)

# Local Fields for $J = 3$

**Figure:** [Field Video!](#)

[Full Caption](#)



**Figure:** In (a),  $J = 1$ ; in (b)  $J = 3$ . For both  $N = 2000$ , and  $K = 6$ .

# Phase distributions for $J = 1$

**Figure:** [Phase Video!](#)

[Full Caption](#)



# Phase distributions for $J = 3$

**Figure:** [Phase Video!](#)

[Full Caption](#)

# Infinite N

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$f(\theta, \omega, u, t)$  = density of oscillators with natural frequency  $\omega$ , interaction vector  $u$ , and phase  $\theta$  at time  $t$ .

$\nu(\theta, \omega, u, t)$  = flow of ditto. Also note:

$$\int f(\theta', \omega', u', t) d\theta' g(\omega') d\omega' \rho(u') du' = 1.$$

# Infinite N: interaction matrix

Much the same as before:

$$J(u, u') := J \sum_{m=1}^K (-1)^m u_m u'_m.$$

The  $u$ 's are random  $K$ -vectors with entries being  $\pm 1$  with equal probability, so

$$\rho(u') = 2^{-K} \sum_v \delta(u' - v),$$

where the sum runs over all the equally likely  $v \in \{\pm 1\}^K$ .

So

$$\dot{\theta}_j = \omega_j + \sum_{k=1}^N J_{jk} \sin(\theta_k - \theta_j)$$

becomes

$$\nu(\theta, \omega, u, t) = \omega + \langle J(u, u') \sin(\theta' - \theta) \rangle, \quad (3)$$

$\langle \cdot \rangle$  denotes integration using the time-dependent measure  $f(\theta', \omega', u', t) d\theta' g(\omega') d\omega' \rho(u') du'$ .

# Infinite N: local fields again

Also,

$$P_j = r_j e^{i\phi_j} := \sum_{k=1}^N J_{jk} e^{i\theta_k},$$

becomes

$$P(u, t) = \langle J(u, u') e^{i\theta'} \rangle.$$

# Infinite N

After some manipulation, this:

$$\nu(\theta, \omega, u, t) = \omega + \langle J(u, u') \sin(\theta' - \theta) \rangle,$$

becomes this:

$$\nu(\theta, \omega, u, t) = \omega + \frac{1}{2i} \left[ e^{-i\theta} P(u, t) - e^{i\theta} P^*(u, t) \right].$$

Ansatz time!

$$f(\theta, \omega, u, t) = \frac{1}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \alpha(\omega, u, t)^n e^{in\theta} + \text{c.c.} \right]$$



# Time to Ott

Following the standard procedure (Which is?), and defining  $a(u, t) := \alpha(-i, u, t)$ , we find

$$\dot{a}(u, t) = -a(u, t) + \frac{P^*(u, t) - a(u, t)^2 P(u, t)}{2}. \quad (4)$$

# Time to Ott

We have

$$\dot{a}(u, t) = -a(u, t) + \frac{P^*(u, t) - a(u, t)^2 P(u, t)}{2},$$

but by using a separate trick (Which is?) we have

$$P(u, t) = \frac{J}{2^K} \sum_{u'} \sum_{m=1}^K (-1)^m u_m u'_m a^*(u', t).$$

By replacing  $P$  with its sum, we get a closed set of  $2^K$  ordinary differential equations for  $a(u, t)$ , one for each possible choice of  $u$ .

# Stability?

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- $J_c$ , the critical coupling, corresponds to the incoherent state no longer being stable.
- **To calculate  $J_c$ , we just need to linearize this ODE and find the first zero eigenvalue.**

The Jacobian is

$$-I + \frac{J}{2^{K+1}}A. \quad (5)$$

Here  $I$  is the  $2^K \times 2^K$  identity matrix and

$$A_{u,v} = \sum_{m=1}^K (-1)^m u_m v_m$$

where the entries of  $A$  have been conveniently indexed by binary strings  $u, v \in \{\pm 1\}^K$ .



# Eigenvalues?

For each integer  $1 \leq n \leq K$  and each binary string  $v \in \{\pm 1\}^K$ , define a vector  $\zeta^{(n)} \in \mathbb{R}^{2^K}$  whose  $v$ th entry is

$$\zeta_v^{(n)} = v_n.$$

# Features of $\zeta^{(n)}$ ?

- The set of all  $K$  distinct  $\zeta^{(n)}$  are orthogonal. (Are they?)
- By using the evenness of  $K$ , then  $A\zeta^{(n)} = (-1)^n 2^K \zeta^{(n)}$ .  
(Really?)

# Other eigens?

Given any  $\eta$  perpendicular to all the  $\zeta^{(n)}$ , one finds  $A\eta = 0$ .

# Stability?

Therefore,  $A$  has exactly three distinct eigenvalues:

- 1  $+2^K$  with multiplicity  $K/2$
- 2  $-2^K$  with multiplicity  $K/2$
- 3  $0$  with multiplicity  $2^K - K$

# Stability?

Since the Jacobian is

$$-I + \frac{J}{2^{K+1}}A,$$

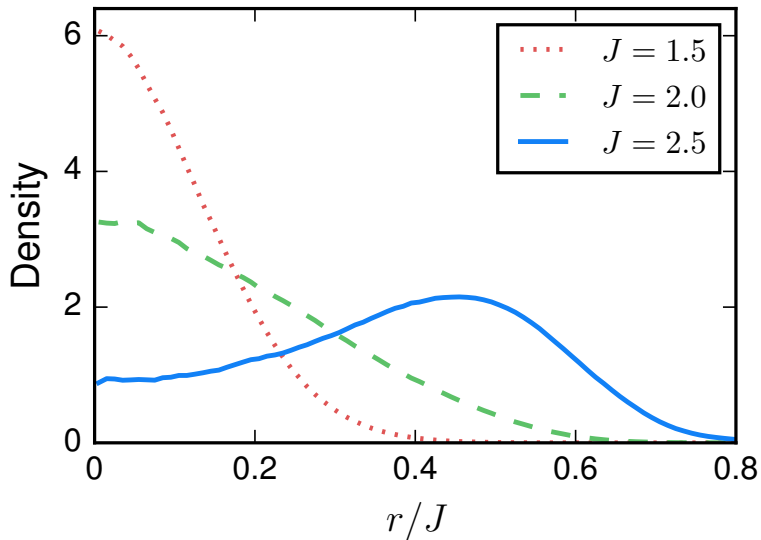
the Jacobian has exactly three distinct eigenvalues:

- 1  $-1 + J/2$  with multiplicity  $K/2$
- 2  $-1 - J/2$  with multiplicity  $K/2$
- 3  $-1$  with multiplicity  $2^K - K$

# THEREFORE

## Theorem

$$J_c = 2. \quad (6)$$





# Numerical checks?

Numerical checks: it's complicated.

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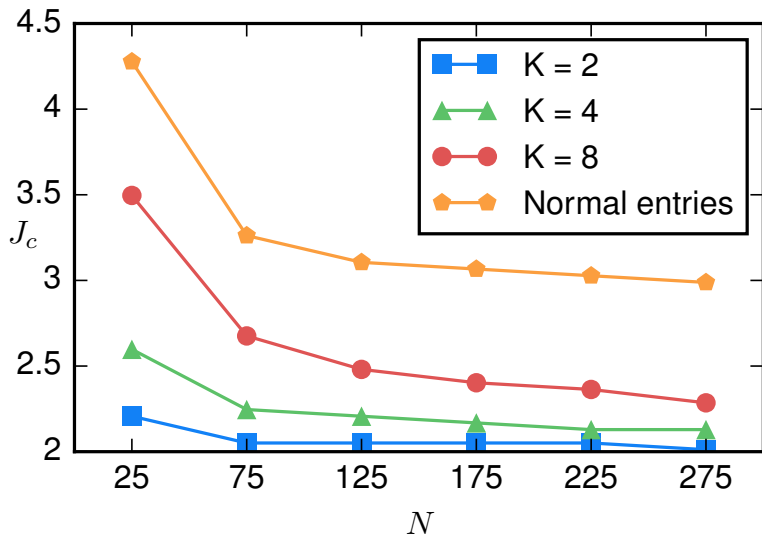
- 1 Find fit to describe volcano transition.
- 2 Parametrically identify the location of transition in fit.
- 3 Find numerical signature of location.
- 4 Prove monotonicity of signature and parameterization.

# Numerical checks: it's complicated.

Expand

- 1 Find fit to describe volcano transition.
- 2 Parametrically identify the location of transition in fit.
- 3 Find numerical signature of location.
- 4 Prove monotonicity of signature and parameterization.
- 5 Use bisection method to identify  $J_c$ .

# Convergence to $J_c = 2$ Expand





Are we a glass yet?

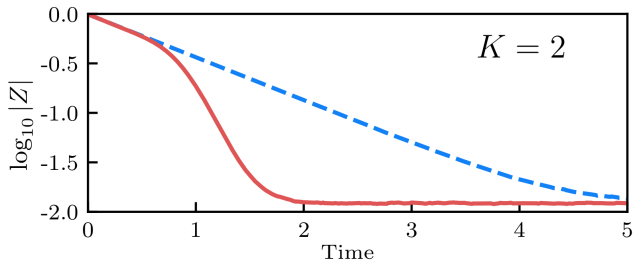
# Are we a glass yet?

A proposed signature feature of oscillator glasses is nonexponential relaxation of the order parameter

$$Z(t) := \sum_{k=1}^N e^{i\theta_k(t)}.$$

No.

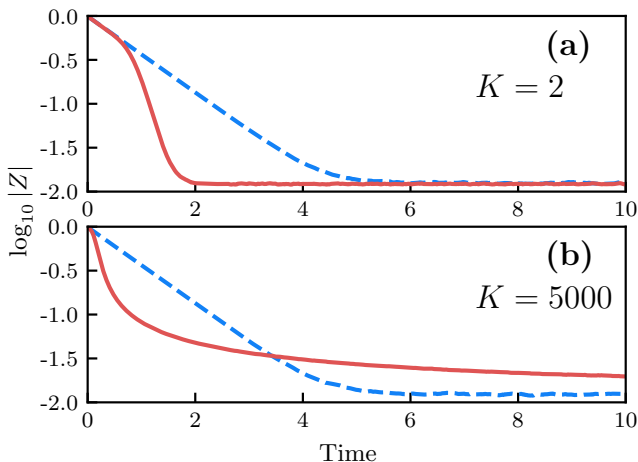
Expand



**Figure:** Dashed blue line has  $J = 0$ . Solid red has  $J = 10$ .  $N = 5000$  for both.

# Is a glass possible?

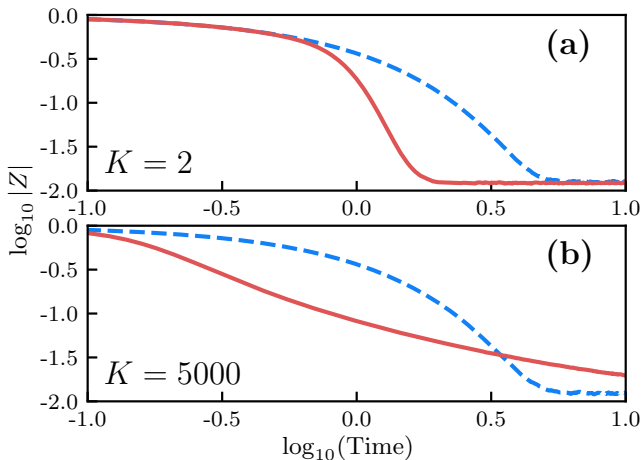
- Our continuum limit gives us  $2^K$  ODEs.
- Therefore, it is only consistent when  $N \gg 2^K$
- What if that doesn't happen?



**Figure:** Dashed blue line has  $J = 0$ . Solid red has  $J = 10$ .  $N = 5000$  for both.

# Algebraic decay?

Expand



**Figure:** Dashed blue line has  $J = 0$ . Solid red has  $J = 10$ .  $N = 5000$  for both.

# Final Summary

- $J_c = 2$ , regardless of the rank of the coupling matrix.
- The volcano transition can occur in absence of a glass transition.
- There is still the possibility of a glass transition in the high  $K$  limit of this model.

# Future directions



# Future directions

Checking the large  $K$  limit for glassy behavior.

# Future directions

Applying results to associative memory models:

$$J_{jk} = \frac{J}{N} \sum_{m=1}^K (1 - \delta_{jk}) e^{\mu_m^{(j)}} e^{\mu_m^{(k)}}$$

with each  $\mu_m^{(j)}$  being uniform on  $0-2\pi$ .

# Future directions

Semi-field parameters are set by

$$F_m = \sum_{k=1}^N u_m^{(k)} e^{i\theta_k}.$$

These have many fun properties, including their relationship to local fields:

$$P_j = \frac{J}{N} \sum_{m=1}^K (-1)^m u_m^{(j)} F_m.$$

They also make for good videos: [Example](#) [Caption](#)

# Selected References



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# Questions?

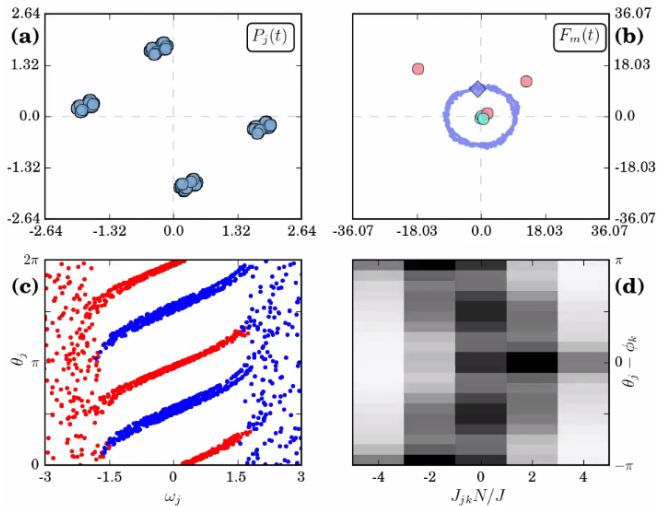


Figure: [Summary Video!](#)

[Full Caption](#)

The only chance of non-trivial dependence is between entries of the same row or column of the coupling matrix, that is, between  $J_{jk}$  and  $J_{jl}$ .

## Yes, they are (II) Return

Take some  $x, y \in \{0, 1, \dots, K\}$ , then

$$\begin{aligned}\mathcal{P} &:= P\left(\frac{N}{J}J_{jk} = 2x - K \text{ and } \frac{N}{J}J_{jl} = 2y - K\right) \\ &= P\left(\sum_{m=1}^K (-1)^m u_m^{(j)} u_m^{(k)} = 2x - K \text{ and } \sum_{m=1}^K (-1)^m u_m^{(j)} u_m^{(l)} = 2y - K\right)\end{aligned}$$

Let  $a_m := (-1)^m u_m^{(j)}$ ,  $b_m := u_m^{(k)}$ ,  $c_m := u_m^{(l)}$ .



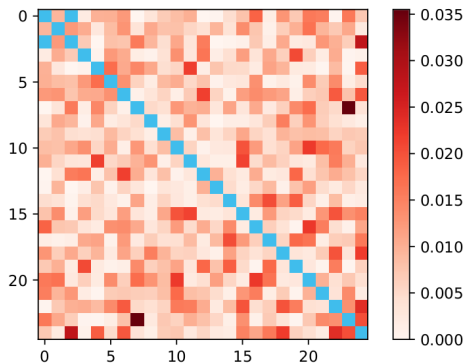
## Yes, they are (III) Return

Let  $a_m := (-1)^m u_m^{(j)}$ ,  $b_m := u_m^{(k)}$ ,  $c_m := u_m^{(l)}$ .

Each can be thought of as a fair  $K$ -long coinflip sequence of  $+1$ 's and  $-1$ 's.

$$\begin{aligned}\mathcal{P} &= P\left(\sum_{m=1}^K a_m b_m = 2x - K \text{ and } \sum_{m=1}^K a_m c_m = 2y - K\right) \\ &= P(\text{coinflip sequence } a \text{ agrees with } b \text{ } x \text{ times and with } c \text{ } y \text{ times}) \\ &= \binom{K}{x} \binom{K}{y} 2^{-2K} \\ &= P\left(\frac{N}{J} J_{jk} = 2x - K\right) P\left(\frac{N}{J} J_{jl} = 2y - K\right) \checkmark\end{aligned}$$

# Yes, they are (IV) Return



**Figure:** Measuring absolute correlation between  $J_{1,2}$  and  $J_{j,k}$  across a coupling matrix of size  $N = 25$  and  $K = 4$ , averaged across  $10^5$  realizations. Blue entries (the diagonal,  $(1,2)$ , and  $(2,1)$ ) have trivial correlations and are ignored.

How does

$$\dot{\theta}_j = \omega_j + \sum_{k=1}^N J_{jk} \sin(\theta_k - \theta_j)$$

become

$$\nu(\theta, \omega, \mathbf{u}, t) = \omega + \langle J(\mathbf{u}, \mathbf{u}') \sin(\theta' - \theta) \rangle? \quad (7)$$

Since

$$J(u, u') := J \sum_{m=1}^K (-1)^m u_m u'_m.$$

we get

$$\dot{\theta}_j = \omega_j + \sum_{k=1}^N \frac{1}{N} J(u^{(j)}, u^{(k)}) \sin(\theta_k - \theta_j).$$

Consider adding a hypothetical oscillator with phase  $\theta$ , natural frequency  $\omega$ , and interaction vector  $u$ . Then its instantaneous frequency would be given by

$$\nu(\theta, \omega, u, t) = \omega + \sum_{k=1}^N \frac{1}{N} J(u, u^{(k)}) \sin(\theta_k - \theta).$$

Sums are just integrals, so

$$\nu(\theta, \omega, u, t) = \omega + \int J(u, u') \sin(\theta_k - \theta) \mu d\theta' d\omega' du',$$

where we have the normalized measure

$$\mu := \frac{1}{N} \sum_{k=1}^N \delta(\theta' - \theta_k) \delta(\omega' - \omega_k) \delta(u' - u^{(k)}).$$

Let's slightly overload our terminology by introducing

$$\hat{\theta}(k, \omega, u, t) = \begin{cases} \theta_k(t) & \text{if } \omega = \omega_k \text{ and } u = u^{(k)}, \\ 0 & \text{else.} \end{cases}$$

Naturally,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \delta(\omega - \omega_k) = g(\omega)$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \delta(u - u^{(k)}) = \rho(u).$$

But now, we define a new measure

$$f(\theta, \omega, u, t) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \delta(\theta - \hat{\theta}(k, \omega, u, t)).$$



Because  $u$  and  $\omega$  are independent, we find that

$$\begin{aligned}\mu &= \frac{1}{N} \sum_{k=1}^N \delta(\theta' - \theta_k) \delta(\omega' - \omega_k) \delta(u' - u_k) \\ &= \frac{1}{N} \sum_{k=1}^N \delta(\theta' - \hat{\theta}(k, \omega, u, t)) \delta(\omega' - \omega_k) \delta(u' - u_k) \\ &\xrightarrow{N \rightarrow \infty} f(\theta, \omega, u, t) g(\omega) \rho(u)\end{aligned}$$

So in the large  $N$  limit, we find that

$$\nu(\theta, \omega, u, t) = \omega + \langle J(u, u') \sin(\theta' - \theta) \rangle,$$

where  $\langle \cdot \rangle$  denotes integration using the time-dependent measure  $f(\theta', \omega', u', t) d\theta' g(\omega') d\omega' \rho(u') du'$ . ✓

How

$$f(\theta, \omega, u, t) = \frac{1}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \alpha(\omega, u, t)^n e^{in\theta} + \text{c.c.} \right]$$

becomes

$$\dot{a}(u, t) = -a(u, t) + \frac{P^*(u, t) - a(u, t)^2 P(u, t)}{2}.$$

Because the total number of oscillators doesn't change, a continuity equation must be obeyed. In particular,

$$-f_t = (f\nu)_\theta.$$

LHS first.

$$\begin{aligned} -f_t &= \partial_t \frac{-1}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \alpha(\omega, u, t)^n e^{in\theta} + \text{c.c.} \right] \\ &= \frac{-1}{2\pi} \sum_{n=1}^{\infty} n \dot{\alpha} \alpha^{n-1} e^{in\theta} + n \dot{\alpha}^* \alpha^{*n-1} e^{-in\theta} \end{aligned}$$

RHS takes more ...

$$f\nu = \frac{1}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \alpha^n e^{in\theta} + \alpha^* n e^{-in\theta} \right] \left[ \omega + \frac{1}{2i} (e^{-i\theta} P - e^{i\theta} P^*) \right].$$

But eventually ...

$$\begin{aligned} f\nu &= \frac{1}{4\pi i} [2\omega i + \alpha P - \alpha^* P^*] \\ &+ \frac{1}{4\pi i} \sum_{n=1}^{\infty} (2\omega i \alpha^n + \alpha^{n+1} P - \alpha^{n-1} P^*) e^{in\theta} \\ &+ \frac{1}{4\pi i} \sum_{n=1}^{\infty} (2\omega i \alpha^{*n} - \alpha^{*n+1} P^* + \alpha^{*n-1} P) e^{-in\theta}. \end{aligned}$$

And so ...

$$(f\nu)_\theta = \frac{1}{4\pi i} \sum_{n=1}^{\infty} (in) (2\omega i \alpha^n + \alpha^{n+1} P - \alpha^{n-1} P^*) e^{in\theta} \\ + \frac{1}{4\pi i} \sum_{n=1}^{\infty} (-in) (2\omega i \alpha^{*n} - \alpha^{*n+1} P^* + \alpha^{*n-1} P) e^{-in\theta}.$$



By setting  $f_t + (f\nu)_\theta = 0$  and setting Fourier coefficients to be equal, then

$$\dot{\alpha} = -\omega\alpha i + \frac{P^* - \alpha^2 P}{2}$$

The calculation of  $P(u, t)$  reveals it only depends on  $\alpha(-i, u, t)$ , so setting  $\omega = -i$  and  $a(u, t) := \alpha(-i, u, t)$ , we finally get

$$\dot{a}(u, t) = -a(u, t) + \frac{P^*(u, t) - a(u, t)^2 P(u, t)}{2}. \checkmark$$

## Another trick explanation (I) Return

How

$$P(u, t) = \langle J(u, u') e^{i\theta'} \rangle$$

becomes

$$P(u, t) = \frac{J}{2^K} \sum_{u'} \sum_{m=1}^K (-1)^m u_m u'_m a^*(u', t).$$

Sub in the ansatz to get

$$\begin{aligned} P(u, t) &= \int J(u, u') e^{i\theta'} f(\theta', \omega', u', t) g(\omega') \rho(u') d\theta' d\omega' du' \\ &= \int \frac{J(u, u')}{2\pi} \left[ e^{i\theta'} + \sum_{n=1}^{\infty} \alpha^n e^{i(n+1)\theta} + \alpha^* n e^{-i(n-1)\theta} \right] \\ &\quad g(\omega') \rho(u') d\theta' d\omega' du'. \end{aligned}$$

Take the  $\theta'$  integral first to get rid of almost everything

$$\begin{aligned} P(u, t) &= \int \frac{J(u, u')}{2\pi} (2\pi\alpha^*) g(\omega') \rho(u') d\omega' du' \\ &= \int J(u, u') \alpha^* g(\omega') \rho(u') d\omega' du'. \end{aligned}$$

## Another trick explanation (IV) Return

Next, we want to take the  $\omega'$  integral. Since we are using a Cauchy distribution, then

$$g(\omega) = \frac{1}{\pi(a + \omega^2)} = \frac{i/(2\pi)}{\omega + i} + \frac{-i/(2\pi)}{\omega - i}.$$

So if  $\alpha$  has the right smoothness and decay, then the integrand has exactly two poles at  $\pm i$ .

So by taking the appropriate contour integral, we get

$$\begin{aligned} P(u, t) &= \int J(u, u') \left( \int \alpha^*(\omega', u', t) g(\omega') d\omega' \right) \rho(u') du' \\ &= \int J(u, u') \left( -2\pi i \frac{i}{2\pi} \alpha^*(-i, u', t) \right) \rho(u') du' \\ &= \int J(u, u') a(u', t) \rho(u') du', \end{aligned}$$

where  $a(u, t) := \alpha(-i, u, t)$ .

From here, just plug in for  $J$ , recalling that

$$\rho(u') = 2^{-K} \sum_v \delta(u' - v)$$

$$\begin{aligned} P(u, t) &= \int J(u, u') a(u', t) \rho(u') du' \\ &= \frac{J}{2^K} \sum_{u'} \sum_{m=1}^K (-1)^m u_m u'_m a^*(u', t). \checkmark \end{aligned}$$



# Why that Jacobian? (I) Return

How

$$\dot{a}(u, t) = -a(u, t) + \frac{P^*(u, t) - a(u, t)^2 P(u, t)}{2}.$$

becomes

$$-I + \frac{J}{2^{K+1}} A.$$

## Why that Jacobian? (II) Return

- We want to linearize around  $a(u, t) = 0$  for all  $u$ .
- $P$  and  $P^*$  are just  $O(a)$ .
- Therefore, we can drop the  $a^2P$  term near 0.

Therefore, near the incoherent state,

$$\begin{aligned}\dot{a}(u, t) &\approx -a(u, t) + \frac{1}{2}P^*(u, t) + O(a^2) \\ &\approx -a(u, t) + \frac{J}{2^{K+1}} \sum_{u'} \sum_{m=1}^K (-1)^m u_m u'_m a(u', t)\end{aligned}$$

## Why that Jacobian? (IV) Return

So the entries of the Jacobian are given by

$$\begin{aligned}\partial_{a(u)} \dot{a}(u) &= -1 + \frac{J}{2^{K+1}} \sum_{m=1}^K (-1)^m u_m u_m \\ &= -1 + \frac{J}{2^{K+1}} \sum_{m=1}^K (-1)^m = -1\end{aligned}$$

on the diagonal, and

$$\partial_{a(v)} \dot{a}(u) = \frac{J}{2^{K+1}} \sum_{m=1}^K (-1)^m u_m v_m$$

of the off-diagonal.

# Why that Jacobian? (V) Return

The most convenient way to index the entries of the Jacobian is by using the vectors  $u, v \in \{+1, -1\}^K$ , AKA the set of all K-long binary strings. So entry  $u, v$  of the Jacobian is

$$-\delta_{u,v} + \frac{J}{2^{K+1}} \sum_{m=1}^K (-1)^m u_m v_m$$

Therefore, the Jacobian is

$$-I + \frac{J}{2^{K+1}}A,$$

where  $I$  is the  $2^K \times 2^K$  identity matrix and

$$A_{u,v} = \sum_{m=1}^K (-1)^m u_m v_m \cdot \checkmark$$

# Yes, they are orthogonal (I) Return

Why is

$$\zeta_v^{(n)} = v_n,$$

orthogonal for all  $n$ ?

# Yes, they are orthogonal (II) Return

Let  $n \neq m$ ,

$$\zeta^{(n)} \zeta^{(m)} = \sum_v v_n v_m.$$



Let  $n \neq m$ , then

$$\begin{aligned}\zeta^{(n)}\zeta^{(m)} &= \sum_v v_n v_m \\ &= |\{\# \text{ times } m \text{ and } n \text{ agree}\}| \\ &\quad - |\{\# \text{ times } m \text{ and } n \text{ disagree}\}| \\ &= 2^{K-2}(1+1) - 2^{K-2}(1+1) \\ &= 0. \checkmark\end{aligned}$$

# Yes, they are eigenvectors (I) Return

Why does

$$A_{uv} = \sum_{m=1}^K (-1)^m u_m v_m$$

have

$$\zeta_v^{(n)} = v_n,$$

as an eigenvector?

## Yes, they are eigenvectors (II) Return

Let's do this directly, and let's fix some  $n \in \{1, 2, \dots, K\}$  and  $u \in \{+1, -1\}^K$ . So

$$\begin{aligned} \left(A\zeta^{(n)}\right)_u &= \sum_v A_{uv} \zeta_v^{(n)} = \sum_v \left( \sum_{m=1}^K (-1)^m u_m v_m \right) v_n \\ &= \sum_{m=1}^K \sum_v (-1)^m u_m v_m v_n \\ &= \sum_v (-1)^n u_n v_n v_n + \sum_{m \neq n} \sum_v (-1)^m u_m v_m v_n \\ &= \text{Term 1} + \text{Term 2} \end{aligned}$$

First term!

$$\begin{aligned}\text{Term 1} &= \sum_v (-1)^n u_n v_n v_n \\ &= (-1)^n u_n \sum_v v_n^2 \\ &= (-1)^n 2^K u_n\end{aligned}$$

The second term therefore becomes

$$\begin{aligned}\text{Term 2} &= \sum_{m \neq n} \sum_{v \in \{\pm 1\}^K} (-1)^m u_m v_m v_n \\ &= \sum_{m \neq n} \sum_{v' \in \{\pm 1\}^{K-1}} u'_m v'_m,\end{aligned}$$

where the  $u'$  and  $v'$  are  $n$ -deleted versions of  $(-1)^m u_m$  and  $v_m v_n$  specifically.

Yes, they are eigenvectors (V) Return

Second term becomes a matter of just counting the number of places two vectors  $u'$  and  $v'$  of length  $K - 1$  disagree, across all possible  $v'$ . So combinatorically,

$$\text{Term 2} = 2 \sum_{w=0}^{K-1} (K - 1 - 2w) \binom{K-1}{w}.$$

But ...

$$\begin{aligned} \text{Term 2} &= 2 \sum_{w=0}^{K-1} (K-1-2w) \binom{K-1}{w} \\ &= 0. \end{aligned}$$

This is an odd function times an even function, so the total sum goes to zero. (Really?)

# Yes, they are eigenvectors (VII) Return

Putting them back together ...

$$\begin{aligned} \left( A\zeta^{(n)} \right)_u &= \text{Term 1} + \text{Term 2} \\ &= (-1)^n 2^K u_n + 0 \\ &= (-1)^n 2^K \zeta_u^{(n)}, \forall u. \end{aligned}$$

Therefore, for all  $n$ , we have

$$A\zeta^{(n)} = (-1)^n 2^K \zeta^{(n)}. \checkmark$$



Why does

$$\eta \perp \zeta^{(n)} \text{ for all } n$$

imply

$$A\eta = 0?$$

Since

$$\eta \perp \zeta^{(n)} \text{ for all } n,$$

then for any  $n = 1, 2, \dots, K$ , we have

$$\eta \zeta^{(n)} = 0.$$

Therefore,

$$\begin{aligned}(A\eta)_u &= \sum_v A_{uv}\eta_u = \sum_v \sum_{m=1}^K (-1)^m u_m v_m \eta_u \\ &= \sum_{m=1}^K (-1)^m u_m \sum_v v_m \eta_u = \sum_{m=1}^K (-1)^m u_m (\eta \zeta^{(n)}) \\ &= \sum_{m=1}^K (-1)^m u_m 0 = 0.\end{aligned}$$

So  $A\eta = 0$ , so it forms a nullspace. ✓.

Measure the location of the peak:

- 1 If the peak is at zero, then we are below the transition
- 2 Otherwise, we are above it.

Measure the location of the peak:

- Radial density noisiest at origin.
- Need to collect and add many pdfs, but ignore most of the data in diagnosis.
- Binning of pdfs will be arbitrary.

Fit the radial distribution of the data to symmetric normals

$$h(r) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(r - \mu)^2}{2\sigma^2}\right) + \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(r + \mu)^2}{2\sigma^2}\right)$$

for  $r \geq 0$ .

The functional form of  $h(r)$  allows us to identify its convexity at the origin easily.

- $\gamma := \mu^2/\sigma^2 < 1$  implies **concave down**.
- $\gamma > 1$  implies **concave up**.

# Measuring $\gamma$ ? Return

We are going to use method of moments on the 2D local field data, with  $M_n$  being the  $n$ 'th moment.



$$M_{+1}M_{-1} = \frac{\pi}{2} \frac{1 + \gamma}{[e^{-\gamma/2} + \sqrt{\pi\gamma/2}\text{Erf}(\sqrt{\gamma/2})]^2}.$$

- The left hand side comes from simulation data. Expand
- The right hand side is monotone in  $\gamma$ . Expand
- Therefore,  $h(r)$  is concave down at the origin (and therefore  $J < J_c$ ) if and only if  $M_{+1}M_{-1} \gtrsim 1.4694$ .

When is the sum of two normals centered at  $\pm\mu$  concave up at the origin?

When is

$$h(r) = \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\mu^2 - r^2}{2\sigma^2}\right) \cosh\left(\frac{\mu r}{\sigma^2}\right),$$

concave up at the origin?

Letting  $\gamma := \mu^2/\sigma^2$  and  $\beta := \mu/\sigma^2$ ,

$$h(r) = \frac{2\beta}{\sqrt{2\pi\gamma}} \exp\left(\frac{-\gamma}{2}\right) \exp\left(\frac{-r^2\beta^2}{2\gamma}\right) \cosh(r\beta).$$

$$\begin{aligned} \partial_r^2 h(r) = & \beta^3 \gamma^{-2} \exp\left(\frac{-\gamma}{2} - \frac{\beta^2 r^2}{2\gamma}\right) \sqrt{\frac{2}{\pi\gamma}} \\ & \times \cosh(\beta r) [-\gamma + \gamma^2 + \beta^2 r^2 - 2\beta\gamma r \tanh(\beta r)] \end{aligned}$$

Evaluating therefore gives

$$\partial_r^2 h(0) = \beta^3 \gamma^{-1} \exp(-\gamma/2) \sqrt{\frac{2}{\pi\gamma}} (\gamma - 1).$$

So  $h$  is concave down whenever  $\gamma < 1$  and concave up whenever  $\gamma > 1$ . ✓

Why does

$$h(r) = \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\mu^2 - r^2}{2\sigma^2}\right) \cosh\left(\frac{\mu r}{\sigma^2}\right),$$

lead to

$$M_{+1}M_{-1} = \frac{\pi}{2} \frac{1 + \gamma}{[e^{-\gamma/2} + \sqrt{\pi\gamma/2}\text{Erf}(\sqrt{\gamma/2})]^2}?$$

Define

$$\mu_n := \int_0^\infty r^n h(r) dr$$



By computation, and using  $\gamma := \mu^2/\sigma^2$  and  $\beta := \mu/\sigma^2$ ,

$$\mu_0 = 1$$

$$\mu_1 = e^{-\gamma/2} \sqrt{\frac{2\gamma}{\pi}} \frac{1}{\beta} + \frac{\gamma}{\beta} \text{Erf} \left( \sqrt{\frac{\gamma}{2}} \right)$$

$$\mu_2 = \frac{\gamma^2 + \gamma}{\beta^2}$$

Notice!

- $h(r)$  is a distribution on the interval  $(0, \infty)$ .
- However, our data comes from a sample of a 2D space.

So the “true” distribution we are sampling is closer to

$$H(x, y) := \frac{h\left(\sqrt{x^2 + y^2}\right)}{2\pi\mu_1},$$

since it has the correct radial behavior, is rotationally symmetric, and is properly normalized on the plane.

So the moments we are numerically observing are

$$\begin{aligned}M_n &= \int \int H(x, y) \left( \sqrt{x^2 + y^2} \right)^n dx dy \\ &= \int \int r^n \frac{h(r)}{2\pi\mu_1} r dr d\theta \\ &= \frac{\mu_{n+1}}{\mu_1}\end{aligned}$$

By earlier computation,

$$M_{-1} = \frac{\mu_0}{\mu_1} = \frac{\beta}{e^{-\gamma/2} \sqrt{\frac{2\gamma}{\pi}} + \gamma \operatorname{Erf} \left( \sqrt{\frac{\gamma}{2}} \right)}$$
$$M_{+1} = \frac{\mu_2}{\mu_1} = \frac{(\gamma^2 + \gamma)/\beta}{e^{-\gamma/2} \sqrt{\frac{2\gamma}{\pi}} + \gamma \operatorname{Erf} \left( \sqrt{\frac{\gamma}{2}} \right)}$$

So, we directly get

$$M_{+1}M_{-1} = \frac{\pi}{2} \frac{1 + \gamma}{[e^{-\gamma/2} + \sqrt{\pi\gamma/2}\text{Erf}(\sqrt{\gamma/2})]^2} \cdot \checkmark$$

# Measuring $M_{+1}M_{-1}$ ? (I) Return

How do we measure  $M_{+1}M_{-1}$ ?

## Measuring $M_{+1}M_{-1}$ ? (II) Return

Estimate individual moments from numerically obtained values of

$$P_j(t) = x_j(t) + iy_j(t) := \sum_{k=1}^N J_{jk} (\cos(\theta_k(t)) + i \sin(\theta_k(t))),$$

by averaging over all  $N$  fields and over the sampling window  $T$ .

$$M_{+1} \approx \frac{1}{NT} \sum_{j=1}^N \sum_{t=1}^T \sqrt{x_j(t)^2 + y_j(t)^2}$$
$$M_{-1} \approx \frac{1}{NT} \sum_{j=1}^N \sum_{t=1}^T \sqrt{x_j(t)^2 + y_j(t)^2}^{-1}$$



## Measuring $M_{+1}M_{-1}$ ? (III) Return

We then estimate their product by averaging these moments across  $L$  independent simulations,

$$E[M_{+1}] = \mu_{+1} \approx \frac{1}{L} \sum_{l=1}^L M_{+1}^{(l)}$$

$$E[M_{-1}] = \mu_{-1} \approx \frac{1}{L} \sum_{l=1}^L M_{-1}^{(l)}$$

$$E[M_{+1}M_{-1}] \approx \frac{1}{L} \sum_{l=1}^L M_{+1}^{(l)} M_{-1}^{(l)}.$$

Note!

$$E[M_{+1}M_{-1}] \neq \mu_{+1}\mu_{-1},$$

because moments aren't independent.

Similarly,

$$\text{Var}(M_{+1}) = \sigma_{+1}^2$$

$$\text{Var}(M_{-1}) = \sigma_{-1}^2$$

can be estimated by online methods, but

$$\text{Var}(M_{+1}M_{-1})$$

is more difficult.

The error on the product is given by

$$\text{Var}(M_{+1}M_{-1}) \approx \frac{1}{L} [\mu_{+1}^2 \sigma_{-1}^2 + \mu_{-1}^2 \sigma_{+1}^2 + 2\mu_{+1}\mu_{-1} \text{Cov}(M_{+1}, M_{-1})] + O\left(\frac{1}{L^2}\right).$$

so this gives us a numerical way to estimate  $M_{+1}M_{-1}$ . ✓

# $M_{+1}M_{-1}$ monotone? (I) Return

How the heck is

$$M_{+1}M_{-1} = \frac{\pi}{2} \frac{1 + \gamma}{[e^{-\gamma/2} + \sqrt{\pi\gamma/2}\text{Erf}(\sqrt{\gamma/2})]^2}$$

montone in  $\gamma$ ?

## $M_{+1}M_{-1}$ monotone? (II) Return

Let's define

$$\frac{\pi}{2f(\gamma)^2} = \frac{\pi}{2} \frac{1 + \gamma}{[e^{-\gamma/2} + \sqrt{\pi\gamma/2}\text{Erf}(\sqrt{\gamma/2})]^2},$$

so this expression is monotone if  $f$  is monotone in  $\gamma$ .

# $M_{+1}M_{-1}$ monotone? (III) Return

Is

$$f(\gamma) = \frac{e^{-\gamma/2} + \sqrt{\pi\gamma/2}\text{Erf}(\sqrt{\gamma/2})}{\sqrt{1+\gamma}},$$

monotone?

Is

$$f'(\gamma) = \frac{\sqrt{\pi/(8\gamma)}}{(1+\gamma)^{3/2}} \left( \text{Erf} \left( \sqrt{\frac{\gamma}{2}} \right) - \sqrt{\frac{2\gamma}{\pi}} e^{-\gamma/2} \right)$$

single signed?

Notice!

$$\operatorname{Erf}\left(\sqrt{\frac{\gamma}{2}}\right) = \int_0^{\sqrt{\gamma/2}} \frac{2}{\sqrt{\pi}} e^{-t^2} dt$$



Also notice!

$$\sqrt{\frac{\gamma}{2}} e^{-\gamma/2} = \int_0^{\sqrt{\gamma/2}} e^{-t^2} - 2t^2 e^{-t^2} dt$$

Therefore!

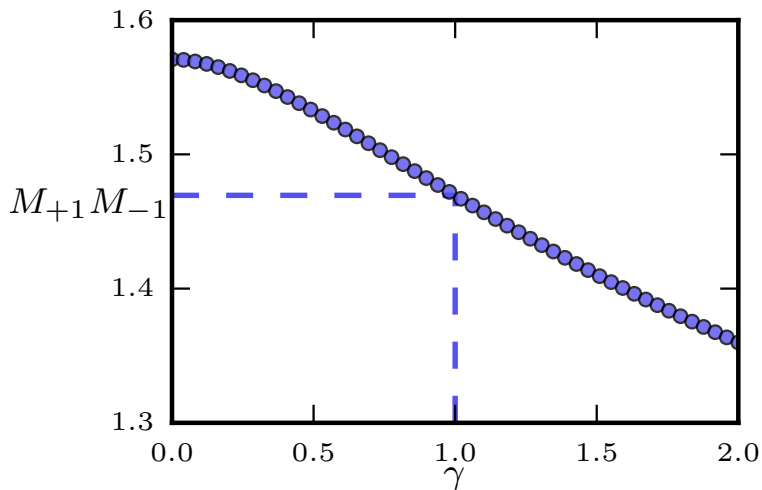
$$f'(\gamma) = \frac{\sqrt{\pi/(8\gamma)}}{(1+\gamma)^{3/2}} \left( \int_0^{\sqrt{\gamma/2}} \frac{2}{\sqrt{\pi}} e^{-t^2} dt \right. \\ \left. - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\gamma/2}} e^{-t^2} - 2t^2 e^{-t^2} dt \right)$$

Therefore!

$$f'(\gamma) = \frac{\sqrt{\pi/(8\gamma)}}{(1+\gamma)^{3/2}} \int_0^{\sqrt{\gamma/2}} \frac{4}{\sqrt{\pi}} t^2 e^{-t^2} dt$$
$$\geq 0.$$

So  $M_{+1}M_{-1}$  is monotone in  $\gamma$ ! ✓

# $M_{+1}M_{-1}$ monotone? (IX) Return



Split the sum in half to get

$$\begin{aligned}\text{Term 2} &= 2 \sum_{w=0}^{K-1} (K-1-2w) \binom{K-1}{w} \\ &= 2 \sum_{w=0}^{K/2-1} (K-1-2w) \binom{K-1}{w} \\ &\quad + 2 \sum_{w=K/2}^{K-1} (K-1-2w) \binom{K-1}{w}.\end{aligned}$$

Letting  $s := K - 1 - w$  gives

$$\begin{aligned} \text{Term 2} = & 2 \sum_{w=0}^{K/2-1} (K - 1 - 2w) \binom{K-1}{w} \\ & + 2 \sum_{s=K/2-1}^0 (-K + 1 + s) \binom{K-1}{K-1-s}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Term 2} &= 2 \sum_{w=0}^{K/2-1} (K-1-2w) \binom{K-1}{w} \\ &\quad - 2 \sum_{s=0}^{K/2-1} (K-1-2s) \binom{K-1}{s} \\ &= 0. \end{aligned}$$

# What does frustration look like?

Return

**Figure:** Each dot represents a distinct  $\theta_j(t)$  over time under using a coupled oscillator model, with zero natural frequencies and random  $\pm 1$  coupling strengths. Here,  $N = 75$ , and we used a fourth-order Runge-Kutta integration with a step size of 0.002 across 2000 recorded steps, and initial phases distributed as a normal about 0. Each frame represents 10 steps.



**Figure:** Distribution of local fields. Each dot represents a distinct  $P_j(t)$  over time under using Daido's setup, with normally distributed coupling strengths and natural frequencies. Here,  $N = 500$ ,  $J = 4$  and we used a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 10 steps.

**Figure:** Distribution of phases versus natural frequencies. Each dot represents a distinct  $\theta_j(t)$  over time under using Daido's setup, with normally distributed coupling strengths and natural frequencies. Here,  $N = 500$ ,  $J = 4$  and we used a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 10 steps.

**Figure:** Distribution of local fields. Each dot represents a distinct  $P_j(t)$  over time under using Daido's setup, with normally distributed coupling strengths and natural frequencies. Here,  $N = 500$ ,  $J = 14$  and we used a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 10 steps.

**Figure:** Distribution of phases versus natural frequencies. Each dot represents a distinct  $\theta_j(t)$  over time under using Daido's setup, with normally distributed coupling strengths and natural frequencies. Here,  $N = 500$ ,  $J = 14$  and we used a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 10 steps.

## New dynamics: $J = 1$ Return

**Figure:** Distribution of local fields. Each dot represents a distinct  $P_j(t)$  over time under using our setup, with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 1$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

**Figure:** Distribution of semi-fields. Each dot represents a distinct  $F_m(t)$  over time under using our setup, with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 1$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

## New dynamics: $J = 1$ Return

**Figure:** Distribution of phases versus natural frequencies. Each dot represents a distinct  $\theta_j(t)$  over time under using our setup, with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 1$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

**Figure:** Density of phase offsets from local fields versus coupling strength across all pairs of oscillators  $j$  and  $k$ . Darker cells correspond to higher densities. This simulation used our setup with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 1$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.



## New dynamics: $J = 1$ Return

**Figure:** Distribution of local fields and semi-fields. Each dot represents a distinct  $P_j(t)$  or  $F_m$  over time under using our setup, with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 1$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

## New dynamics: $J = 1$ Return

**Figure:** Distribution of phases versus natural frequencies. This simulation used our setup with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 1$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

## New dynamics: $J = 1$ Return

**Figure:** Combination of all other videos. This simulation used our setup with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 1$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

## New dynamics: $J = 3$ Return

**Figure:** Distribution of local fields. Each dot represents a distinct  $P_j(t)$  over time under using our setup, with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 3$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

**Figure:** Distribution of semi-fields. Each dot represents a distinct  $F_m(t)$  over time under using our setup, with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 3$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

## New dynamics: $J = 3$ Return

**Figure:** Distribution of phases versus natural frequencies. Each dot represents a distinct  $\theta_j(t)$  over time under using our setup, with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 3$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

## New dynamics: $J = 3$ Return

**Figure:** Density of phase offsets from local fields versus coupling strength across all pairs of oscillators  $j$  and  $k$ . Darker cells correspond to higher densities. This simulation used our setup with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 3$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

## New dynamics: $J = 3$ Return

**Figure:** Distribution of local fields and semi-fields. Each dot represents a distinct  $P_j(t)$  or  $F_m(t)$  over time under using our setup, with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 3$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.



## New dynamics: $J = 3$ Return

**Figure:** Distribution of phases versus natural frequencies. This simulation used our setup with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 3$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

## New dynamics: $J = 3$ Return

**Figure:** Combination of all other videos. This simulation used our setup with an interaction-vector based coupling and Cauchy distributed natural frequencies. Here,  $N = 2500$ ,  $K = 6$ ,  $J = 3$  and using a fourth-order Runge-Kutta integration with a step size of 0.01, 3600 recorded steps, and uniformly random initial phases. Each frame represents 100 steps.

**Figure:** Radial distribution of local fields. Each curve represents the averaged density over 500 simulations of Eq. (1), using  $N = 250$ ,  $K = 4$ , fourth-order Runge-Kutta integration with a step size of 0.01, 1000 transient steps, 2000 recorded steps, and uniformly random initial phases.

**Figure:** Radial distribution of local fields. Each curve represents the averaged density over 500 simulations of Eq. (1), using  $N = 250$ ,  $K = 4$ , fourth-order Runge-Kutta integration with a step size of 0.01, 1000 transient steps, 2000 recorded steps, and uniformly random initial phases.

**Figure:** Oscillator phase distributions below and above the volcano transition. In (a),  $J = 1$ ; in (b),  $J = 3$ . Each panel shows results for simulations of  $N = 2000$  and  $K = 6$ ; other parameters as in Fig. 31. (a) Below the volcano transition, the system is incoherent. (b) Above the volcano transition, phase-locked clusters appear.

## Convergence to $J_c = 2$ Return

**Figure:** Critical value  $J_c$  versus  $N$  and  $K$ . Each value of  $J_c$  was estimated by using a bisection method on the value of  $M_{+1}M_{-1}$  to achieve an accuracy of  $\lesssim 0.02$ . For each  $J$  we sample  $J_{jk}$  at least 100 times, simulate Eq. (1), evaluate  $M_{+1}M_{-1}$ , and keep track of the running standard deviation of these products. If the current value of  $M_{+1}M_{-1}$  is more than 1.5 standard deviations from 1.4694, the bisection continues; otherwise further simulations are run, up to a maximum of  $10^5$  simulations. Each simulation consists of 1000 transient steps followed by 2000 recorded steps of a fourth-order Runge Kutta integration with a step size of 0.01, with initial phases all set to 0.

**Figure:** Log-log plot for the decay of the order parameter  $Z(t)$ . Each curve is the average of 750 numerical integrations of Eq. (1) for  $N = 5000$  oscillators starting from the in-phase state ( $\theta_j = 0$  for all  $j$ ) and run for 1000 steps with a step size of 0.01. Solid curves show coupled systems with  $J = 10$ ; dashed curves show uncoupled systems with  $J = 0$  for which the order parameter decays exponentially:  $Z(t) = e^{-t}$ . Simulation ran in the low-rank regime:  $K \ll \log_2(N)$ . For  $K = 2$ ,  $Z(t)$  decays exponentially down to the noise floor. Exponential decay is expected in this regime because the dynamics of Eq. (1) are well approximated by the low-dimensional system (4).

**Figure:** Log-log plot for the decay of the order parameter  $Z(t)$ . Each curve is the average of 750 numerical integrations of Eq. (1) for  $N = 5000$  oscillators starting from the in-phase state ( $\theta_j = 0$  for all  $j$ ) and run for 1000 steps with a step size of 0.01. Solid curves show coupled systems with  $J = 10$ ; dashed curves show uncoupled systems with  $J = 0$  for which the order parameter decays exponentially:  $Z(t) = e^{-t}$ . (a) Low-rank regime:  $K \ll \log_2(N)$ . For  $K = 2$ ,  $Z(t)$  decays exponentially down to the noise floor. Exponential decay is expected in this regime because the dynamics of Eq. (1) are well approximated by the low-dimensional system (4). (b) High-rank regime:  $K = N = 5000$ . When  $K = O(N)$  and  $J > J_c$ , the relaxation of  $Z$  slows markedly, resembling the algebraic decay in glass.



**Figure:** Log-log plot for the decay of the order parameter  $Z(t)$ . Each curve is the average of 750 numerical integrations of Eq. (1) for  $N = 5000$  oscillators starting from the in-phase state ( $\theta_j = 0$  for all  $j$ ) and run for 1000 steps with a step size of 0.01. Solid curves show coupled systems with  $J = 10$ ; dashed curves show uncoupled systems with  $J = 0$  for which the order parameter decays exponentially:  $Z(t) = e^{-t}$ . (a) Low-rank regime:  $K \ll \log_2(N)$ . For  $K = 2$ ,  $Z(t)$  decays exponentially down to the noise floor. Exponential decay is expected in this regime because the dynamics of Eq. (1) are well approximated by the low-dimensional system (4). (b) High-rank regime:  $K = N = 5000$ . When  $K = O(N)$  and  $J > J_c$ , the relaxation of  $Z$  slows markedly, resembling the algebraic decay in glass.